Geometry-of-numbers over number fields and the density of ADE families of curves having squarefree discriminant

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Abstract

For families of curves arising from a Dynkin diagram of type ADE, we show that the density of such curves having squarefree discriminant is equal to the product of local densities. We do so using the framework of Thorne and Laga's PhD theses and geometry-of-numbers techniques developed by Bhargava, here expanded over number fields.

Contents

1 Introduction

In this paper, we aim to determine the density of curves in certain families that have squarefree discriminant. Over $\mathbb{Q}$, this was done in [Oll23], and in this article we will generalise the methods in said paper to a general number field F. For completeness and convenience, this article will be self-contained, in the sense that it will develop the results in [Oll23] from scratch.

Our methods will resemble the geometry-of-numbers developed by Bhargava and his collaborators. The main idea is that many arithmetic objects of interest can be parametrised by the rational or integral orbits of a certain representation (G, V): in this situation, Bhargava's geometry-of-numbers methods allow to count these integral orbits of V, which consequently provides information on the desired arithmetic objects that would be otherwise difficult to obtain. This idea has led to many impressive results in number theory; see [Bha14a] or [Ho13] for an overview.

The present paper is inspired by the recent paper [BSW22a] by Bhargava, Shankar and Wang, in which they compute the density of monic integral polynomials of a given degree that have squarefree discriminant. The main technical difficulty is to bound the tail estimate of polynomials having discriminant "weakly divisible" by the square a large prime (this notion will be defined later). They do so using the representation of $G = \mathrm{SO}_n$ on the space V of $n \times n$ symmetric matrices. By relating polynomials with discriminant divisible by p^2 for a large p to certain integral orbits of the representation (G, V), they get the desired result using the aforementioned geometry-of-numbers techniques. Similar methods were used in [BSW22b] in the non-monic case with a different representation, and also in [BH22] for certain families of elliptic curves (in particular, their F_2 case essentially corresponds to our D_4 case).

A key observation, which motivates our results, is that the representation studied in [BSW22a] arises as a particular case of the more general families of representations studied in [Tho13]. Using the framework of

Vinberg theory, Thorne found that given a simply laced Dynkin diagram, we can naturally associate to it a family of curves and a coregular representation (G, V), where the rational orbits of the representation are related to the arithmetic of the curves in the family. These results have been used, implicitly and explicitly, to study the size of 2-Selmer groups of the Jacobians of these curves, see [BG13; SW18; Sha18; Tho15; RT18; Lag22b] for some particular cases. Later, Laga unified, reproved and extended all these results in [Lag22a] in a uniform way.

Our aim is to compute the density of curves having squarefree discriminant in these families of ADE curves. We will do so by reinterpreting the methods in [BSW22a] in the language of [Tho13] and [Lag22a]. Given that we prove our results over a general number field F, this presents an additional challenge, in the sense that most of the literature on the required geometry-of-numbers works over $\mathbb{Q}$, and the translation to the number field case is not necessarily immediate. Taking [BSW15] and [BSW] as a point of reference, we develop the techniques that we need in geometry-of-numbers over a number field.

Let $\mathcal{D}$ be a Dynkin diagram of type A, D, E, and let F be a number field. In Section 2.1, we will construct a representation (G, V) associated to $\mathcal{D}$, and in Section 2.3 we will construct a family of curves $C \to B$. Here, B is isomorphic to the Geometric Invariant Theory (GIT) quotient $V /\!\!/ G := \operatorname{Spec} F[V]^G$. We see that B can be identified with an affine space, and we write $B = \operatorname{Spec} F[p_{d_1}, \ldots, p_{d_r}]$. The group $\mathbb{G}_m$ acts on B by $\lambda \cdot p_{d_i} = \lambda^{d_i} p_{d_i}$. We want to define a sensible notion of *height* for elements of $b \in B(F)$. In Section 4.1, for $b \in B(F)$ we will define

$$\operatorname{ht}(b) := (NI_b) \prod_{v \in M_\infty} \sup\left(|p_{d_1}(b)|_v^{1/d_1}, \ldots, |p_{d_r}(b)|_v^{1/d_r}\right),$$

where M_∞ denotes the set of archimedean places of F and I_b is the ideal $I_b = \{a \in F \mid a^{d_i} p_{d_i}(b) \in \mathcal{O}_F,\ \forall i\}$. This height is $\mathbb{G}_m(F)$-invariant by the product formula. Further, we can see that the number of elements of $\mathbb{G}_m(F)\backslash B(F)$ having bounded height is finite; see [Den98, Theorem A] for a more precise count. Throughout the paper, we will fix a subset $\Sigma \subset B(\mathcal{O}_F)$ which is a fundamental domain for the action of $\mathbb{G}_m(F)$ on $B(F)$: we will construct it in Section 4.1.

Denote by C_b the preimage of a given $b \in B$ under the map $C \to B$; it will be a curve of the form given by Table 2. The main result of this paper concerns the density of squarefree values of the discriminant $\Delta(C_b)$ of the curve (or equivalently, the discriminant $\Delta(b)$ defined in Section 2.1). A definition for the discriminant of a plane curve can be found in [Sut19, §2], for instance. We remark that in our definition of discriminant, we assume that it is an polynomial in multiple variables and coefficients in $\mathcal{O}_F$, normalised so that the coefficients have common divisors (for instance, the usual discriminant for elliptic curves contains a factor of 16: we omit it in our case).

For a prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$, we will denote the completion of $\mathcal{O}_F$ with respect to $\mathfrak{p}$ by $\mathcal{O}_\mathfrak{p}$. We will further set $F_\mathfrak{p}$ to be the field of fractions of $\mathcal{O}_\mathfrak{p}$ and $k_\mathfrak{p}$ to be the corresponding residue field. Now, consider the set

$$B'(\mathcal{O}_\mathfrak{p}) = \{b \in B(\mathcal{O}_\mathfrak{p}) \mid v_\mathfrak{p}(p_{d_i}(b)) < d_i \text{ for some } i\}.$$

Every element of $B(F_\mathfrak{p})$ is $\mathbb{G}_m(F_\mathfrak{p})$-conjugate to an element of $B'(\mathcal{O}_\mathfrak{p})$, and this element is unique up to the $\mathbb{G}_m(\mathcal{O}_\mathfrak{p})$-action. For a given $b \in B(F_\mathfrak{p})$, we say that $\Delta(b)$ is *squarefree at* $\mathfrak{p}$ if, for any element $b' \in B'(\mathcal{O}_\mathfrak{p})$ which is $\mathbb{G}_m(F_\mathfrak{p})$-conjugate to b, the discriminant $\Delta(b')$ is squarefree as an element of $\mathcal{O}_\mathfrak{p}$. For $b \in B(F)$, we say that $\Delta(b)$ is *squarefree* if $\Delta(b)$ is squarefree at $\mathfrak{p}$ for all finite primes $\mathfrak{p}$. Note that the property of "being squarefree" does not change with the action of $\mathbb{G}_m(F)$.

Our result is related to the $\mathfrak{p}$-adic density of these squarefree values: we will denote by $\rho(\mathcal{D}_\mathfrak{p})$ the local density at $\mathfrak{p}$ of curves in the family $C \to B$ having discriminant indivisible by $\mathfrak{p}^2$ in $F_\mathfrak{p}$; this is obtained by taking all the (finitely many) elements in $b \in B(F_\mathfrak{p}/\mathfrak{p}^2 F_\mathfrak{p})$ and counting the proportion of them that have non-zero discriminant in $F_\mathfrak{p}/\mathfrak{p}^2 F_\mathfrak{p}$. We note that under our assumptions on the discriminant none of the local densities vanish; this can be checked with a case-by-case computation.

Theorem 1.1. *We have*

$$\lim_{X\to\infty} \frac{\#\{b \in \mathbb{G}_m(F)\backslash B(F) \mid \Delta(b) \text{ is squarefree, } \operatorname{ht}(b) < X\}}{\#\{b \in \mathbb{G}_m(F)\backslash B(F) \mid \operatorname{ht}(b) < X\}} = \prod_{\mathfrak{p}} \rho(\mathcal{D}_{\mathfrak{p}}).$$

To prove Theorem 1.1, we need to obtain a tail estimate to show that "not too many" $b \in B(\mathcal{O}_F)$ have discriminant divisible by I^2 for squarefree ideals I of large norm. Let $\mathfrak{p}$ be a prime ideal. A key observation in [BSW22a] is to separate those b with $\mathfrak{p}^2$ dividing $\Delta(b)$ in two separate cases:

1. If $\mathfrak{p}^2|\Delta(b+pc)$ for all $p \in \mathfrak{p}$ and $c \in B(\mathcal{O}_F)$, we say $\mathfrak{p}^2$ *strongly divides* $\Delta(b)$ (in other words, $\mathfrak{p}^2$ divides $\Delta(b)$ for "mod $\mathfrak{p}$ reasons").
2. If there exists $p \in \mathfrak{p}$ and $c \in B(\mathcal{O}_F)$ such that $\mathfrak{p}^2 \nmid \Delta(b+pc)$, we say $\mathfrak{p}^2$ *weakly divides* $\Delta(b)$ (in other words, $\mathfrak{p}^2$ divides $\Delta(b)$ for "mod $\mathfrak{p}^2$ reasons").

Analogously, for a squarefree ideal $I \subset \mathcal{O}_F$, we will say that I^2 strongly (resp. weakly) divides $\Delta(b)$ if every prime ideal $\mathfrak{p} \mid I$ strongly (resp. weakly) divides $\Delta(b)$. We will let $\mathcal{W}_I^{(1)}, \mathcal{W}_I^{(2)}$ denote the set of $b \in B(\mathcal{O}_F)$ whose discriminant is strongly (resp. weakly) divisible by I^2. We want to prove tail estimates for $\mathcal{W}_I^{(1)}, \mathcal{W}_I^{(2)}$ separately. Our argument for the weakly divisible case will require us to avoid finitely many primes: more precisely, in Section 3 we will define an element $N_{bad} \in \mathcal{O}_F$ which will be divisible by all these "bad primes".

Theorem 1.2. *There exists a constant $\delta > 0$ such that for any positive real number M we have:*

$$\sum_{\substack{I \text{ squarefree} \\ NI > M}} \#\{b \in \mathbb{G}_m(F)\backslash \mathcal{W}_I^{(1)} \mid \operatorname{ht}(b) < X\} = O_\varepsilon\left(\frac{X^{\dim V+\varepsilon}}{M}\right) + O_\varepsilon\left(X^{\dim V - 1+\varepsilon}\right),$$

$$\sum_{\substack{I \text{ squarefree} \\ NI > M \\ (I, N_{bad})=1}} \#\{b \in \mathbb{G}_m(F)\backslash \mathcal{W}_I^{(2)} \mid \operatorname{ht}(b) < X\} = O_\varepsilon\left(\frac{X^{\dim V+\varepsilon}}{M}\right) + O\left(X^{\dim V - \delta}\right).$$

The implied constants are independent of X and M.

The strongly divisible case will follow from the use of the Ekedahl sieve: see Section 6.2 for a discussion. Hence, we will spend most of the paper dealing with the weakly divisible case.

We start in Section 2, where we develop the necessary background and introducing our objects of interest, most importantly the representation (G, V) coming from Vinberg theory and the associated family of curves $C \to B$. The main step in the proof of Theorem 1.2 is done in Section 3, where given an element $b \in \mathcal{W}_I^{(2)}$, we construct a special integral orbit in V, whose elements have invariant b. We additionally consider a distinguished subspace $W_0 \subset V$, and we define a Q-invariant for the elements of W_0. Then, we will see that the elements in the constructed orbit have large Q-invariant when they intersect W_0 (which happens always except for a negligible amount of times by cutting-off-the-cusp arguments). This construction is the analogue of [BSW22a, §2.2, §3.2]; we give a more detailed comparison at the end of Section 3.

In Section 4, we set up the main tools that we will require in reduction theory. This includes an extended discussion about heights, as well as a construction of a suitable "box-shaped" fundamental domain for the action of an arithmetic subgroup Γ of $G(F)$ over $G(F_\infty)$, where $F_\infty = \prod_{v\in M_\infty} F_v$. Then, in Section 5, we compute a precise asymptotic for the number of reducible orbits of bounded height. This is a necessary step in the argument, since the proof of the main results requires a power-saving asymptotic on the elements with big stabiliser (Proposition 6.1), which will require the results in Section 5. We remark that part of the argument relies on extensive case-by-case computations: some of them are carried out in

Section 5.4, while some others take place implicitly in the proof of Proposition 5.3. Finally, in Section 6 we conclude the proof of the main results. In Section 6.2 we prove Theorem 1.2, and in Section 6.3 we deduce Theorem 1.1 using a squarefree sieve.

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1.1 Notation

We recap the most important bits of notation in this section. Most of it has already been introduced or will be introduced in the future, but is included here for the convenience of the reader.

Throughout, we will work with a fixed number field F. We will denote its ring of integers by $\mathcal{O}_F$. For a finite prime $\mathfrak{p}$ of F, we will denote by $\mathcal{O}_\mathfrak{p}$ the completion of $\mathcal{O}_F$ with respect to $\mathfrak{p}$, $F_\mathfrak{p}$ its field of fractions and $k_\mathfrak{p}$ its residue field. We will denote the set of infinite places of F by M_∞, and for any $v \in M_\infty$, we will denote by F_v the completion of F with respect to v. For an ideal $I \subset \mathcal{O}_F$, we will denote by the norm of the ideal by $NI = \#(\mathcal{O}_F/I)$.

We will also denote $F_\infty = \prod_{v\in M_\infty} F_v$, and for $x = (x_v)_v \in F_\infty$ we will denote

$$|x| := \prod_v |x_v|_v,$$

where $|x_v|_v$ denotes the norm in F_v given by $|x_v|_v = N_{\mathbb{C}/F_v}(x_v)$. For $x \in F$, we will denote by $|x|$ the norm of F as an element of F_∞.

Given a split semisimple group H, we will consider a natural representation (G, V), where G is a suitable subgroup of H. Inside G, we will fix a split torus T, and a Borel subgroup P containing T, corresponding to the negative roots of $\Phi(G, T)$. We will also fix $B := V /\!\!/ G = \operatorname{Spec} F[p_{d_1}, \dots, p_{d_r}]$, where p_{d_i} are polynomials of degree i, on which $\mathbb{G}_m(F)$ acts upon by $\lambda \cdot p_{d_i} = \lambda^{d_i} p_{d_i}$. We will also fix a Kostant section $\kappa \cdot B \to V$, which will be a section of the invariant map $\pi \colon V \to B$.

We will also fix the unipotent radical N of the Borel P, a maximal compact subgroup K of $G(F_\infty)$ and the subgroup A of $T(F_\infty)$ consisting of $t = (t_1, \dots, t_k)$ such that $(t_i)_v \in \mathbb{R}_{>0}$ for every i and every place $v \in M_\infty$.

2 Preliminaries

In this section, we introduce our representation (G, V) of interest, together with some of its basic properties. We do so mostly following [Tho13, §2] and [Lag22a, §3].

2.1 Vinberg representations

Let F be a number field, let $H_{\mathbb{Q}}$ be a split adjoint simple group of type A, D, E over $\mathbb{Q}$, and consider H to be the base change of $H_{\mathbb{Q}}$ to F. We assume H is equipped with a pinning $(T_H, P_H, \{X_\alpha\})$, meaning:

- $T_H \subset H$ is a split maximal torus defined over $\mathbb{Q}$ (determining a root system Φ_H).
- $P_H \subset H$ is a Borel subgroup containing T_H (determining a root basis $S_H \subset \Phi_H$).
- X_α is a generator for $\mathfrak{h}_\alpha$ for each $\alpha \in S_H$.

Let $W = N_H(T_H)/T_H$ be the Weyl group of Φ_H, and let $\mathcal{D}$ be the Dynkin diagram of H. Then, we have the following exact sequences:

$$0 \longrightarrow H \longrightarrow \mathrm{Aut}(H) \longrightarrow \mathrm{Aut}(\mathcal{D}) \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow W \longrightarrow \mathrm{Aut}(\Phi_H) \longrightarrow \mathrm{Aut}(\mathcal{D}) \longrightarrow 0 \tag{2}$$

The subgroup $\mathrm{Aut}(H, T_H, P_H, \{X_\alpha\}) \subset \mathrm{Aut}(H)$ of automorphisms of H preserving the pinning determines a splitting of (1). Then, we can define $\vartheta \in \mathrm{Aut}(H)$ as the unique element in $(T_H, P_H, \{X_\alpha\})$ such that its image in $\mathrm{Aut}(\mathcal{D})$ under (1) coincides with the image of $-1 \in \mathrm{Aut}(\Phi_H)$ under (2). Writing $\check{\rho}$ for the sum of fundamental coweights with respect to S_H, we define

$$\theta := \vartheta \circ \mathrm{Ad}(\check{\rho}(-1)) = \mathrm{Ad}(\check{\rho}(-1)) \circ \vartheta.$$

The map θ defines an involution of H, and so $d\theta$ defines an involution of the Lie algebra $\mathfrak{h}$. By considering ± 1 eigenspaces, we obtain a $\mathbb{Z}/2\mathbb{Z}$-grading

$$\mathfrak{h} = \mathfrak{h}(0) \oplus \mathfrak{h}(1),$$

where $[\mathfrak{h}(i), \mathfrak{h}(j)] \subset \mathfrak{h}(i+j)$. We define $G = (H^\theta)^\circ$ and $V = \mathfrak{h}(1)$, which means that V is a representation of G by restriction of the adjoint representation. Moreover, we have $\mathrm{Lie}(G) = \mathfrak{h}(0)$.

We have the following basic result [Pan05, Theorem 1.1] on the GIT quotient $B := V /\!\!/ G = \mathrm{Spec}\, F[V]^G$.

Theorem 2.1. *Let $\mathfrak{c} \subset V$ be a Cartan subspace. Then, $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{h}$, and the map $N_G(\mathfrak{c}) \to W_{\mathfrak{c}} := N_H(\mathfrak{c})/Z_H(\mathfrak{c})$ is surjective. Therefore, the canonical inclusions $\mathfrak{c} \subset V \subset \mathfrak{h}$ induce isomorphisms*

$$\mathfrak{c} /\!\!/ W_{\mathfrak{c}} \cong V /\!\!/ G \cong \mathfrak{h} /\!\!/ H.$$

In particular, all these quotients are isomorphic to a finite-dimensional affine space.

For any field k of characteristic zero, we can define the *discriminant polynomial* $\Delta \in k[\mathfrak{h}]^H$ as the image of $\prod_{\alpha \in \Phi_T} \alpha$ under the isomorphism $k[\mathfrak{t}]^W \xrightarrow{\sim} k[\mathfrak{h}]^H$. The discriminant can also be regarded as a polynomial in $k[B]$ through the isomorphism $k[\mathfrak{h}]^H \cong k[V]^G = k[B]$. We can relate the discriminant to one-parameter subgroups, which we now introduce. If $\lambda \colon \mathbb{G}_m \to G_k$ is a homomorphism, there exists a decomposition $V = \sum_{i \in \mathbb{Z}} V_i$, where $V_i := \{v \in V(k) \mid \lambda(t)v = t^i v,\ \forall t \in \mathbb{G}_m(k)\}$. Every vector $v \in V(k)$ can be written as $v = \sum v_i$, where $v_i \in V_i$; we call the integers i with $v_i \neq 0$ the *weights* of v. Finally, we recall that an element $v \in \mathfrak{h}$ is *regular* if its centraliser has minimal dimension.

Proposition 2.2. *Let $k/\mathbb{Q}$ be a field, and let $v \in V(k)$. The following are equivalent:*

1. *v is regular semisimple.*
2. *$\Delta(v) \neq 0$.*

3. *For every non-trivial homomorphism $\lambda\colon \mathbb{G}_m \to G_{k^s}$, v has a positive weight with respect to λ.*

Proof. The reasoning is the same as in [RT18, Corollary 2.4]. □

We remark that the Vinberg representation (G, V) can be identified explictly. For the reader's convenience, we reproduce the explicit description written in [Lag22a, §3.2] in Table 1. We refer the reader to *loc. cit.* for the precise meaning of some of these symbols.

Type	G	V
A_{2n}	SO_{2n+1}	$\mathrm{Sym}^2(2n+1)_0$
A_{2n+1}	PSO_{2n+2}	$\mathrm{Sym}^2(2n+2)_0$
D_{2n} $(n \geq 2)$	$\mathrm{SO}_{2n} \times \mathrm{SO}_{2n} / \Delta(\mu_2)$	$2n \boxtimes 2n$
D_{2n+1} $(n \geq 2)$	$\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n+1}$	$(2n+1) \boxtimes (2n+1)$
E_6	PSp_8	$\wedge_0^4 8$
E_7	SL_8 / μ_4	$\wedge^4 8$
E_8	Spin_{16}/μ_2	half spin

Table 1: Explicit description of each representation

2.2 Restricted roots

In the previous section we considered the root system Φ_H of H, but we will also need to work with the root system of G. By [Ric82, Lemma 5.1], we have that $T := T_H^{\vartheta}$ is a split maximal torus of G. We will compare the root systems Φ_H and $\Phi_G = \Phi(G, T)$ in a similar fashion to [Tho15, §2.3].

Write Φ_H/ϑ for the orbits of ϑ on Φ_H, where ϑ is the pinned automorphism defined in the previous section.

Lemma 2.3. *1. The map $X^*(T_H) \to X^*(T)$ is surjective, and the group G is adjoint. In particular, $X^*(T)$ is spanned by Φ_G.*

2. Let $\alpha, \beta \in \Phi_H$. Then, the image of α in $X^*(T)$ is non-zero, and α, β have the same image if and only if either $\alpha = \beta$ or $\alpha = \vartheta(\beta)$.

Proof. The fixed group T is connected and contains regular elements of T_H by [Ree10, Lemma 3.1]. The group G has trivial center by [Ree10, §3.8]. For the second part, see [Ree10, §3.3]. □

Hence, we can identify Φ_H/ϑ with its image in $X^*(T)$. We note that $\vartheta = 1$ if and only if -1 is an element of the Weyl group $W(H, T_H)$; in this case Φ_H/ϑ coincides with Φ_H.

We can write the following decomposition:

$$\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{a \in \Phi_H/\vartheta} \mathfrak{h}_a,$$

with $\mathfrak{t} = \mathfrak{t}^{\theta} \oplus V_0$ and $\mathfrak{h}_a = \mathfrak{g}_a \oplus V_a$, according to the θ-grading. We have a decomposition

$$V = V_0 \oplus \bigoplus_{a \in \Phi_H/\vartheta} V_a.$$

For a given $a \in \Phi_H/\vartheta$ there are three cases to distinguish, according to the value of $s = (-1)^{\langle \alpha, \check{\rho} \rangle}$:

1. $a = \{\alpha\}$ and $s = 1$. Then, $V_a = 0$ and $\mathfrak{g}_\alpha$ is spanned by X_α.
2. $a = \{\alpha\}$ and $s = -1$. Then, V_a is spanned by X_α and $\mathfrak{g}_\alpha = 0$.
3. $a = \{\alpha, \vartheta(\alpha)\}$, with $\alpha \neq \vartheta(\alpha)$. Then, V_a is spanned by $X_\alpha - sX_{\vartheta(\alpha)}$ and $\mathfrak{g}_\alpha$ is spanned by $X_\alpha + sX_{\vartheta(\alpha)}$.

We note that ϑ preserves the height of a root α with respect to the basis S_H. Therefore, it will make sense to define the *height* of a root $a \in \Phi/\vartheta$ as the height of any element in $\vartheta^{-1}(a)$.

2.3 Transverse slices over $V /\!\!/ G$

In this section, we present some remarkable properties of the map $\pi\colon V \to B$, where we recall that $B := V /\!\!/ G$ is the GIT quotient.

Definition 2.4. An $\mathfrak{sl}_2$*-triple* of $\mathfrak{h}$ is a triple (e, h, f) of non-zero elements of $\mathfrak{h}$ satisfying

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Moreover, we say this $\mathfrak{sl}_2$-triple is *normal* if $e, f \in \mathfrak{h}(1)$ and $h \in \mathfrak{h}(0)$.

Theorem 2.5 (Graded Jacobson-Morozov)**.** *Every non-zero nilpotent element $e \in \mathfrak{h}(1)$ is contained in a normal $\mathfrak{sl}_2$-triple. If e is also regular, then it is contained in a unique normal $\mathfrak{sl}_2$-triple.*

Proof. The first part of the statement is [Tho13, Lemma 2.17], and the second part follows from [Tho13, Lemma 2.14]. □

Definition 2.6. Let r be the rank of $\mathfrak{h}$. We say an element $x \in \mathfrak{h}$ is *subregular* if $\dim \mathfrak{z}_{\mathfrak{h}}(x) = r + 2$.

Subregular elements in V exist by [Tho13, Proposition 2.27]. Let $e \in V$ be such an element, and fix a normal $\mathfrak{sl}_2$-triple (e, h, f) using Theorem 2.5. Let $C = e + \mathfrak{z}_V(f)$, and consider the natural morphism $\varphi\colon C \to B$.

Theorem 2.7. *1. The geometric fibres of φ are reduced connected curves. For $b \in B(F)$, the corresponding curve C_b is smooth if and only if $\Delta(b) \neq 0$.*

2. The central fibre $\varphi^{-1}(0)$ has a unique singular point which is a simple singularity of type A_n, D_n, E_n, coinciding with the type of H.

3. We can choose coordinates $p_{d_1}, \ldots, p_{d_r}$ in B, with p_{d_i} being homogeneous of degree d_i, and coordinates $(x, y, p_{d_1}, \ldots, p_{d_r})$ on C such that $C \to B$ is given by Table 2.

Proof. See [Tho13, Theorem 3.8]. □

Our choice of pinning in Section 2.1 determines a natural choice of a regular nilpotent element, namely $e_0 = \sum_{\alpha \in S_H} X_\alpha \in V(F)$. Let (e_0, h_0, f_0) be its associated normal $\mathfrak{sl}_2$-triple by Theorem 2.5. We define the affine linear subspace $\kappa_{e_0} := (e_0 + \mathfrak{z}_{\mathfrak{h}}(f_0)) \cap V$ as the *Kostant section* associated to e_0. Whenever e_0 is understood, we will just denote the Kostant section by κ.

Theorem 2.8. *The composition $\kappa \hookrightarrow V \to B$ is an isomorphism, and every element of κ is regular.*

Proof. See [Tho13, Lemma 3.5]. □

Definition 2.9. Let L/F be a field and let $v \in V(L)$. We say v is L*-reducible* if $\Delta(v) = 0$ or if v is $G(L)$-conjugate to some Kostant section, and F*-irreducible* otherwise.

We will typically refer to F-(ir)reducible elements simply as (ir)reducible. We note that if L is algebraically closed, then all elements of V are reducible, by [Lag22a, Proposition 2.11].

Type	Curve	# Marked points
A_{2n}	$y^2 = x^{2n+1} + p_2 x^{2n-1} + \cdots + p_{2n+1}$	1
A_{2n+1}	$y^2 = x^{2n+2} + p_2 x^{2n} + \cdots + p_{2n+2}$	2
D_{2n} $(n \geq 2)$	$y(xy + p_{2n}) = x^{2n-1} + p_2 x^{2n-2} + \cdots + p_{4n-2}$	3
D_{2n+1} $(n \geq 2)$	$y(xy + p_{2n+1}) = x^{2n} + p_2 x^{2n-1} + \cdots + p_{4n}$	2
E_6	$y^3 = x^4 + (p_2 x^2 + p_5 x + p_8)y + (p_6 x^2 + p_9 x + p_{12})$	1
E_7	$y^3 = x^3 y + p_{10} x^2 + x(p_2 y^2 + p_8 y + p_{14}) + p_6 y^2 + p_{12} y + p_{18}$	2
E_8	$y^3 = x^5 + (p_2 x^3 + p_8 x^2 + p_{14} x + p_{20})y + (p_{12} x^3 + p_{18} x^2 + p_{24} x + p_{30})$	1

Table 2: Families of curves

2.4 Integral structures

So far, we have considered our objects of interest over the fields $\mathbb{Q}$ and F, but for our purposes it will be crucial to define integral structures for G and V.

We start by considering structures over $\mathbb{Z}$. The structure of G over $\mathbb{Z}$ comes from the general classification of split reductive groups over any non-empty scheme S: namely, every root datum is isomorphic to the root datum of a split reductive S-group (see [Con14, Theorem 6.1.16]). By considering the root datum $\Phi(G, T)$ studied in Section 2.2 and the scheme $S = \operatorname{Spec} \mathbb{Z}$, we get a split reductive group $\underline{G}_{\mathbb{Z}}$ defined over $\mathbb{Z}$, such that its base change to $\mathbb{Q}$ coincides with G. By [Ric82, Lemma 5.1], we know that T is a maximal split torus of G, and that $P := P_H^\theta$ is a Borel subgroup of G containing T. We also get integral structures for $\underline{T}_{\mathbb{Z}}$ and $\underline{P}_{\mathbb{Z}}$ inside of $\underline{G}_{\mathbb{Z}}$. We get $\mathcal{O}_F$-structures by base-changing to $\mathcal{O}_F$: set $\underline{G} := \underline{G}_{\mathbb{Z}} \otimes \mathcal{O}_F$ and analogously $\underline{T} := \underline{T}_{\mathbb{Z}} \otimes \mathcal{O}_F$, $\underline{P} := \underline{P}_{\mathbb{Z}} \otimes \mathcal{O}_F$.

For *any* linear algebraic group G defined over F, we recall that its class group is

$$\operatorname{cl}(G) = \Big(\prod_{\mathfrak{p} \notin M_\infty} G(\mathcal{O}_{\mathfrak{p}}) \Big) \backslash G(\mathbb{A}_{F,f}) / G(F).$$

Proposition 2.10. *Every linear algebraic group G has finite class group.*

Proof. See [Bor63, Theorem 5.1]. □

To obtain a $\mathbb{Z}$-structure for $V_{\mathbb{Q}}$, we consider $\mathfrak{h}_{\mathbb{Q}}$ as a semisimple $G_{\mathbb{Q}}$-module over $\mathbb{Q}$ via the restriction of the adjoint representation. This $G_{\mathbb{Q}}$-module splits into a sum of simple $G_{\mathbb{Q}}$-modules:

$$\mathfrak{h} = (\oplus_{i=1}^r V_i) \oplus (\oplus_{i=1}^s \mathfrak{g}_i),$$

where $\oplus V_i = V_{\mathbb{Q}}$ and $\oplus \mathfrak{g}_i = \mathfrak{g}_{\mathbb{Q}}$, since both subspaces are $G_{\mathbb{Q}}$-invariant. For each of these irreducible representations, we can choose highest weight vectors $v_i \in V_i$ and $w_i \in \mathfrak{g}_i$, and we then consider

$$\underline{V}_i := \operatorname{Dist}(\underline{G}_{\mathbb{Z}}) v_i, \quad \underline{\mathfrak{g}}_i := \operatorname{Dist}(\underline{G}_{\mathbb{Z}}) w_i,$$

where $\operatorname{Dist}(\underline{G}_{\mathbb{Z}})$ the algebra of distributions of $\underline{G}_{\mathbb{Z}}$ (see [Jan07, I.7.7]). Analogously to [Jan07, II.8.3], we have that $V_i = \mathbb{Q} \otimes_{\mathbb{Z}} \underline{V}_i$, $\mathfrak{g}_i = \mathbb{Q} \otimes_{\mathbb{Z}} \underline{\mathfrak{g}}_i$ and that $\underline{V}_{\mathbb{Z}} := \oplus \underline{V}_i$ is a $\underline{G}_{\mathbb{Z}}$-stable lattice inside $V_{\mathbb{Q}}$. As before, set $\underline{V} := \underline{V}_{\mathbb{Z}} \otimes \mathcal{O}_F$, which is a $\underline{G}_{\mathcal{O}_F}$-stable lattice inside V. By scaling the highest weight vectors if necessary, we will assume that $E \in \underline{V}(\mathcal{O}_F)$.

We can also consider an integral structure $\underline{B}$ on B. We can take the polynomials $p_{d_1}, \ldots, p_{d_r} \in F[V]^G$ determined in Section 2.3 and rescale them using the $\mathbb{G}_m$-action $t \cdot p_{d_i} = t^{d_i} p_{d_i}$ to make them lie in $\mathcal{O}_F[\underline{V}]^{\underline{G}}$. We let $\underline{B} := \operatorname{Spec} \mathcal{O}_F[p_{d_1}, \ldots, p_{d_r}]$ and write $\pi \colon \underline{V} \to \underline{B}$ for the corresponding morphism. We may additionally assume that the discriminant Δ defined in Section 2.1 lies in $\mathcal{O}_F[\underline{V}]^{\underline{G}}$, where the coefficients of Δ in $\mathcal{O}_F[p_{d_1}, \ldots, p_{d_r}]$ may be assumed to not have a common divisor.

The following lemma will be convenient to us in the future (cf. [Tho15, Lemma 2.8]):

Lemma 2.11. *There exists a non-zero $N_0 \in \mathcal{O}_F$ such that for all primes $\mathfrak{p}$ and for all $b \in \underline{B}(\mathcal{O}_\mathfrak{p})$ we have $N_0 \cdot \kappa_b \in \underline{V}(\mathcal{O}_\mathfrak{p})$.*

Our arguments in Section 3 will implicitly rely on integral geometric properties of the representation (G, V). In there, we will need to avoid finitely many primes, or more precisely to work over $S = \operatorname{Spec} \mathcal{O}_F[1/N']$ for a suitable $N' \in \mathcal{O}_F$. By combining the previous lemma and the spreading out properties in [Lag22a, §7.2], we get:

Proposition 2.12. *There exists a non-zero element $N' \in \mathcal{O}_F$ such that:*

1. *For every $b \in \underline{B}(\mathcal{O}_F)$, the corresponding Kostant section κ_b is $G(F)$-conjugate to an element in $\frac{1}{N'}\underline{V}(\mathcal{O}_F)$.*
2. *N' is admissible in the sense of [Lag22a, §7.2].*

Fix an element $N' \in \mathcal{O}_F$ satisfying the conclusions of Proposition 2.12. From now on, we will simplify notation by dropping the underline notation for the objects defined over $\mathcal{O}_F$, and just refer to $\underline{G}, \underline{V}, \dots$ as $G, V, \dots$ by abuse of notation.

To end this section, we consider some further integral properties of the Kostant section. In Section 2.3, we considered κ defined over F, and now we will consider some of its properties over $\mathcal{O}_\mathfrak{p}$. Consider the decomposition

$$\mathfrak{h} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{h}_j$$

according to the height of the roots. If P_H^- is the negative Borel subgroup of H, N_H^- is its unipotent radical and $\mathfrak{p}^-$ and $\mathfrak{n}^-$ are their respective Lie algebras, we have $\mathfrak{p}^- = \bigoplus_{j \leq 0} \mathfrak{h}_j$, $\mathfrak{n}^- = \bigoplus_{j<0} \mathfrak{h}_j$ and $[E, \mathfrak{h}_j] \subset \mathfrak{h}_{j+1}$.

Theorem 2.13. *Let R be a ring in which N' is invertible. Then:*

1. *$[E, \mathfrak{n}_R^-]$ has a complement in $\mathfrak{p}_R^-$ of rank $rk_R\, \mathfrak{p}_R^- - rk_R\, \mathfrak{n}_R^-$; call it Ξ.*
2. *The action map $N_H^- \times (E + \Xi) \to E + \mathfrak{p}^-$ is an isomorphism over R.*
3. *Both maps in the composition $E + \Xi \to (E + \mathfrak{p}^-) /\!\!/ N_H^- \to \mathfrak{h} /\!\!/ H$ are isomorphisms over R.*

Proof. See [AFV18, §2.3]. □

Remark 2.14. If R is a field of characteristic zero, then Ξ can be taken to be $\mathfrak{z}_\mathfrak{h}(f_0)$ and $E + \Xi$ is the same as the Kostant section considered in Section 2.3. We will abuse notation by referring to both the Kostant section defined in Section 2.3 and the section in Theorem 2.13 by κ.

3 Constructing orbits

Given an element $b \in B(\mathcal{O}_F)$ with discriminant weakly divisible by I^2 for a squarefree ideal I avoiding certain bad primes, we will show how to construct a special integral orbit in V in a way that "remembers I".

We will start with some techincal results. Assume we have a connected reductive group L over a field k, together with an involution ξ. As in Section 2.1, the Lie algebra $\mathfrak{l}$ decomposes as $\mathfrak{l} = \mathfrak{l}(0) \oplus \mathfrak{l}(1)$, according to the ± 1 eigenspaces of $d\xi$. We also write L_0 for the connected component of the fixed group L^ξ.

Definition 3.1. Let k be algebraically closed. We say a vector $v \in \mathfrak{l}(1)$ is *stable* if the L_0-orbit of v is closed and its stabiliser $Z_{L_0}(v)$ is finite. We say $(L_0, \mathfrak{l}(1))$ is stable if it contains stable vectors. If k is not necessarily algebraically closed, we say $(L_0, \mathfrak{l}(1))$ is stable if $(L_{0,k^s}, \mathfrak{l}(1)_{k^s})$ is.

By [Tho16, Proposition 1.9], the involution θ defined in Section 2.1 is a stable involution, i.e. (G, V) is stable.

We now prove the analogue of [RT21, Lemma 2.3]: the proof is very similar and is reproduced for convenience.

Lemma 3.2. *Let S be a $\mathcal{O}_F[1/N']$-scheme. Let (L, ξ), (L', ξ') be two pairs, each consisting of a reductive group over S whose geometric fibres are adjoint semisimple of type A_1, together with a stable involution. Then for any $s \in S$ there exists an étale morphism $S' \to S$ with image containing s and an isomorphism $L_{S'} \to L'_{S'}$ intertwining $\xi_{S'}$ and $\xi'_{S'}$.*

Proof. We are working étale locally on S, so we can assume that $L = L'$ and that they are both split reductive groups. Let T denote the scheme of elements $l \in L$ such that $\mathrm{Ad}(l) \circ \xi = \xi'$: by [Con14, Proposition 2.1.2], T is a closed subscheme of L that is smooth over S. Since a surjective smooth morphism has sections étale locally, it is sufficient to show that $T \to S$ is surjective. Moreover, we can assume that $S = \operatorname{Spec} k$ for an algebraically closed field k, since the formation of T commutes with base change.

Let $A, A' \subset H$ be maximal tori on which ξ, ξ' act as an automorphism of order 2. By the conjugacy of maximal tori, we can assume that $A = A'$ and that ξ, ξ' define the (unique) element of order 2 in the Weyl group. Write $\xi = a\xi'$ for some $a \in A(k)$. Writing $a = b^2$ for some $b \in A(k)$, we have $\xi = b \cdot b \cdot \xi' = b \cdot \xi' \cdot b^{-1}$. The conclusion is that ξ and ξ' are $H(k)$-conjugate (in fact, $A(k)$-conjugate), which completes the proof. ☐

The following lemma is the key technical input in our proof. We remark the the first part was already implicitly proven in the proof of [Lag22a, Theorem 7.16].

Lemma 3.3. *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_F$ not dividing N', where N' is as in Proposition 2.12.*

1. *Let $b \in B(\mathcal{O}_\mathfrak{p})$ be an element with $\mathrm{ord}_\mathfrak{p} \Delta(b) = 1$, where $\mathrm{ord}_\mathfrak{p} \colon F_\mathfrak{p}^* \to \mathbb{Z}$ is the usual normalized valuation. Let $v \in V(\mathcal{O}_\mathfrak{p})$ with $\pi(v) = b$. Then, the reduction mod $\mathfrak{p}$ of v in $V(k_\mathfrak{p})$ is regular.*
2. *Let $b \in B(\mathcal{O}_\mathfrak{p})$ be an element with discriminant weakly divisible by $\mathfrak{p}^2$. Then, there exists $g_\mathfrak{p} \in G(F_\mathfrak{p}) \setminus G(\mathcal{O}_\mathfrak{p})$ such that $g_\mathfrak{p} \cdot \kappa_b \in V(\mathcal{O}_\mathfrak{p})$.*

Proof. Let $v_{k_\mathfrak{p}} = x_s + x_n$ be the Jordan decomposition of the reduction of v in $k_\mathfrak{p}$. Then, we have a decomposition $\mathfrak{h}_{k_\mathfrak{p}} = \mathfrak{h}_{0,k_\mathfrak{p}} \oplus \mathfrak{h}_{1,k_\mathfrak{p}}$, where $\mathfrak{h}_{0,k_\mathfrak{p}} = \mathfrak{z}_\mathfrak{h}(x_s)$ and $\mathfrak{h}_{1,k_\mathfrak{p}} = \mathrm{image}(\mathrm{Ad}(x_s))$. By Hensel's lemma, this decomposition lifts to $\mathfrak{h}_{\mathcal{O}_\mathfrak{p}} = \mathfrak{h}_{0,\mathcal{O}_\mathfrak{p}} \oplus \mathfrak{h}_{1,\mathcal{O}_\mathfrak{p}}$, with $\mathrm{ad}(v)$ acting topologically nilpotently in $\mathfrak{h}_{0,\mathcal{O}_\mathfrak{p}}$ and invertibly in $\mathfrak{h}_{1,\mathcal{O}_\mathfrak{p}}$. As explained in the proof of [Lag22b, Lemma 4.19], there is a unique closed subgroup $L \subset H_{\mathcal{O}_\mathfrak{p}}$ which is smooth over $\mathcal{O}_\mathfrak{p}$ with connected fibres and with Lie algebra $\mathfrak{h}_{0,\mathcal{O}_\mathfrak{p}}$.

For the first part of the lemma, assume that $\mathcal{O}_\mathfrak{p}$ has uniformiser t. We are free to replace $\mathcal{O}_\mathfrak{p}$ for a complete discrete valuation ring R with the same uniformiser t, containing $\mathcal{O}_\mathfrak{p}$ and with algebraically closed residue field k. In this case, the spreading out properties in [Lag22a, §7.2] guarantee that the derived group of L is of type A_1. Since the restriction of θ restricts to a stable involution in L by [Tho13, Lemma 2.5], Lemma 3.2 guarantees that there exists an isomorphism $\mathfrak{h}_{0,R}^{der} \cong \mathfrak{sl}_{2,R}$ intertwining the action of θ on $\mathfrak{h}_{0,R}^{der}$ with the action of $\xi = \mathrm{Ad}(\mathrm{diag}(1,-1))$ on $\mathfrak{sl}_{2,R}$. To show that v_k is regular is equivalent to showing that the nilpotent part x_n is regular in $\mathfrak{h}_{0,k}^{der}$. The elements v_k and x_n have the same projection in $\mathfrak{h}_{0,k}^{der}$, and given that $v \in \mathfrak{h}_{0,R}^{der,d\theta=-1}$, its image in $\mathfrak{sl}_{2,R}$ is of the form

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix},$$

with $\operatorname{ord}_R(ab) = 1$ by the spreading out properties in [Lag22a, §7.2]. In particular, exactly one of a, b is non-zero when reduced to k, and hence x_n is regular in $\mathfrak{h}_{0,k}^{der}$, as wanted.

For the second part, we return to the case $R = \mathcal{O}_\mathfrak{p}$. There exists $b' \in B(\mathcal{O}_\mathfrak{p})$ such that $\operatorname{ord}_\mathfrak{p} \Delta(b+tb') = 1$, where t is a uniformiser of $\mathcal{O}_\mathfrak{p}$. Because the Kostant section is algebraic, we see that $\kappa_b - \kappa_{b+tb'} \in tV(\mathcal{O}_\mathfrak{p})$, and given that $\kappa_{b+tb'}$ is regular in $V(k_\mathfrak{p})$, we get that κ_b is regular in $V(k_\mathfrak{p})$. In particular, this means that the nilpotent part x_n is a regular nilpotent in $\mathfrak{h}_{0,k_\mathfrak{p}}^{der}$. We now claim that we have an isomorphism $\mathfrak{h}_{0,\mathcal{O}_\mathfrak{p}}^{der} \cong \mathfrak{sl}_{2,\mathcal{O}_\mathfrak{p}}$ intertwining θ and the previously defined ξ and sending the regular nilpotent x_n to the matrix

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

of $\mathfrak{sl}_{2,k_\mathfrak{p}}$. Indeed, consider the $\mathcal{O}_\mathfrak{p}$-scheme $X = \operatorname{Isom}((L/Z(L), \theta), (\mathrm{PGL}_2, \xi))$, consisting of isomorphisms between $L/Z(L)$ and PGL_2 that intertwine the θ and ξ-actions. Using Lemma 3.2, we see that étale-locally, X is isomorphic to $\operatorname{Aut}(\mathrm{PGL}_2, \xi)$; in particular, it is a smooth scheme over $\mathcal{O}_\mathfrak{p}$. By Hensel's lemma [Gro67, Théorème 18.5.17], to show that X has a $\mathcal{O}_\mathfrak{p}$-point it is sufficient to show that it has an $k_\mathfrak{p}$-point.

Now, consider the $k_\mathfrak{p}$-scheme $Y = \operatorname{Isom}((L/Z(L)_{k_\mathfrak{p}}, \theta, x_n), (\mathrm{PGL}_2, \xi, e))$ of isomorphisms preserving the θ and ξ-actions which send x_n to e: it is a subscheme of $X_{k_\mathfrak{p}}$. Again by Lemma 3.2, Y is étale locally of the form $\operatorname{Aut}(\mathrm{PGL}_2, \xi, e)$, since PGL_2^ξ acts transitively on the regular nilpotents of $\mathfrak{sl}_2^{d\xi=-1}$ for any field of characteristic not dividing N'. In particular, we see that Y is an $\operatorname{Aut}(\mathrm{PGL}_2, \xi, e)$-torsor. In this situation, to see that $Y(k_\mathfrak{p})$ is non-empty and hence that $X(\mathcal{O}_\mathfrak{p})$ is non-empty, it will suffice to see that $\operatorname{Aut}(\mathrm{PGL}_2, \xi, e) = \operatorname{Spec} k_\mathfrak{p}$. This follows from the elementary computation of the stabiliser of e under PGL_2^ξ, which can be seen to be trivial over any field.

In conclusion, $X(\mathcal{O}_\mathfrak{p})$ is non-empty, meaning that there is an isomorphism $\mathfrak{h}_{0,\mathcal{O}_\mathfrak{p}}^{der} \cong \mathfrak{sl}_{2,\mathcal{O}_\mathfrak{p}}$ respecting θ and ξ, and we can make it so that the projection of v in $\mathfrak{sl}_{2,\mathcal{O}_\mathfrak{p}}$ is an element of the form

$$\begin{pmatrix} 0 & a \\ bt^2 & 0 \end{pmatrix},$$

with $a, b \in \mathcal{O}_\mathfrak{p}$ and $a \in 1 + t\mathcal{O}_\mathfrak{p}$. Moreover, there exists a morphism $\varphi \colon \mathrm{SL}_2 \to L_{F_\mathfrak{p}}^{der}$ inducing the given isomorphism $\mathfrak{h}_{0,F_\mathfrak{p}}^{der} \cong \mathfrak{sl}_{2,F_\mathfrak{p}}$, since SL_2 is simply connected. The morphism φ necessarily respects the grading, and induces a map $\mathrm{SL}_2(F_\mathfrak{p}) \to L^{der}(F_\mathfrak{p})$ on the $F_\mathfrak{p}$-points. Consider the matrix $g_\mathfrak{p} = \varphi(\operatorname{diag}(t, t^{-1}))$: it satisfies the conditions of the lemma, and so we are done. □

If I^2 divides $\Delta(b)$ weakly, we get for each prime $\mathfrak{p}$ dividing I an element $g_\mathfrak{p}$ as above. Now, consider the adelic element $g' \in G(\mathbb{A}_{F,f})$ defined by $g_\mathfrak{p}$ at every $\mathfrak{p}$ dividing I and by 1 at every other prime. Recall that by Proposition 2.10, we can fix elements $\beta_1, \dots, \beta_k$ such that

$$G(\mathbb{A}_{F,f}) = \coprod_{i=1}^{k} \left(\prod_{\mathfrak{p}} G(\mathcal{O}_\mathfrak{p}) \right) \beta_i G(F).$$

Thus, we can write $g' = (h_\mathfrak{p})_\mathfrak{p} \beta_i g_I$ for some $h_\mathfrak{p} \in G(\mathcal{O}_\mathfrak{p})$ and $g_I \in G(F)$. The element $g_I \cdot \kappa_b$ does not necessarily lie in $V(\mathcal{O}_F)$, but rather in

$$V_{\beta_i} = V(F) \cap \beta_i^{-1} \prod_{\mathfrak{p}} V(\mathcal{O}_\mathfrak{p}).$$

We see that V_{β_i} is a lattice inside $V(F)$, commensurable with $V(\mathcal{O}_F)$. In particular, the elements of the different $V_{\beta_1}, \dots, V_{\beta_k}$ are all contained in $\frac{1}{N_{bad}} V(\mathcal{O}_F)$ for some $N_{bad} \in \mathcal{O}_F$. In what follows, we will only consider primes $\mathfrak{p}$ that don't divide this element N_{bad}, and in particular we can assume that $N' \mid N_{bad}$ The choice of g_I is not uniquely determined, it is unique up to the action of

$$G_{\beta_i} = G(F) \cap \beta_i^{-1} \left(\prod_{\mathfrak{p}} G(\mathcal{O}_{\mathfrak{p}}) \right) \beta_i.$$

This is a subgroup of $G(F)$ which is commensurable with $G(\mathcal{O}_F)$. We further define the distinguished subspace $W_0 \subset V$ as

$$W_0 := \bigoplus_{\substack{a \in \Phi_H/\vartheta \\ \mathrm{ht}(a) \le 1}} V_a,$$

where the notation is as in Section 2.2. We write an element $v \in W_0(F)$ as $v = \sum_{\mathrm{ht}(\alpha)=1} v_\alpha X_\alpha + \sum_{\mathrm{ht}(\beta) \le 0} v_\beta X_\beta$, where each X_α, X_β generates each root space V_α, V_β and $v_\alpha, v_\beta \in F$. Then, we can define the *Q-invariant* of $v \in W_0$ as $Q(v) = \prod_{\mathrm{ht}(\alpha)=1} v_\alpha$. For a squarefree ideal $I \subset \mathcal{O}_F$, set $G_I = G(F) \setminus \cup_{\mathfrak{p}|I} (G(\mathcal{O}_{\mathfrak{p}}) \cap G(F))$, that is, the elements of $G(F)$ "having denominators in I". Now, define:

$$W_{i,M} := \{ v \in V_{\beta_i} \mid v = g_I \kappa_b,\ I \text{ squarefree},\ I \text{ coprime to } N_{bad},\ NI > M,\ g_I \in G_I,\ b \in B(\mathcal{O}_K) \}.$$

The main result of the section is the following:

Proposition 3.4. *Let $b \in B(\mathcal{O}_F)$, and assume that* $\mathrm{Stab}_{G(F)} \kappa_b = \{e\}$.

1. *Let I be a squarefree ideal coprime to N_{bad} and satisfying $NI > M$. If I^2 weakly divides $\Delta(b)$, then $W_{i,M} \cap \pi^{-1}(b)$ is non-empty.*
2. *If $w \in W_{i,M} \cap W_0$, then $\prod_{v \in M_\infty} |Q(w)|_v > M$.*

Proof. We start by proving the first item. In the above discussion, we constructed an element $g_I \in G(F)$ such that $g_i \kappa_b \in V_{\beta_i}$. Given that I is coprime to N_{bad}, the construction shows that $g_I \in G_I$, as otherwise we would have $(\beta_i)_{\mathfrak{p}} \notin G(\mathcal{O}_{\mathfrak{p}})$ for some prime $\mathfrak{p} \mid I$, a contradiction.

We now prove the second item. Assume $w = g_I \kappa_b$ for suitable I and b. It suffices to show that $\mathrm{ord}_{\mathfrak{p}} Q(w) \ge 1$: in particular, it suffices to assume that $I = \mathfrak{p}$. Given that H is an adjoint group, there exists a $t \in T_H(F)$ that makes all the height-one coefficients of $t\kappa_b$ be equal to one, and in this case we see that actually $t \in T(F)$. By Theorem 2.13, there exists a unique $\gamma \in N_H^-(F)$ such that $\gamma t \kappa_b = w$; by taking θ-invariants in the isomorphisms of Theorem 2.13, we see that $\gamma \in N_H^{-,\theta}(F)$. Since the stabiliser is trivial, we see that $g = \gamma t$, or in other words that $g \in P(F) \setminus (P(\mathcal{O}_{\mathfrak{p}}) \cap P(F))$.

Assume, for the sake of contradiction, that $Q(v)$ is invertible in $\mathcal{O}_{\mathfrak{p}}$, so that all the height-one coefficients of w are invertible. Then, there exists a $t' \in T(\mathcal{O}_{\mathfrak{p}})$ making all the height-one coefficients of $t'w$ be equal to one, and by Theorem 2.13, there exists at most one element γ' in $N^-(\mathcal{O}_{\mathfrak{p}})$ such that $\gamma' t' \kappa_b = w$. This would imply that $g \in P(\mathcal{O}_{\mathfrak{p}})$, a contradiction. □

Example 3.5. Our construction is inspired by the construction in [BSW22a, Sections 2.2 and 3.2] for the case A_n, for $F = \mathbb{Q}$. In that case, $C \to B$ corresponds to the family of hyperelliptic curves $y^2 = f(x)$, where $f(x)$ has degree $n+1$ (there is a slight difference between this paper and [BSW22a], in that we consider $f(x)$ without an x^n term while they consider polynomials with a possibly non-zero linear term; we ignore this difference for now). The main goal of [BSW22a, Sections 2.2 and 3.2] is to construct an embedding

$$\sigma_m \colon \mathcal{W}_2^{(m)} \to \frac{1}{4} W_0(\mathbb{Z}) \subset \frac{1}{4} V(\mathbb{Z}),$$

where $\sigma_m(f)$ has characteristic polynomial f and $Q(\sigma_m(f)) = m$. By taking the usual pinning in SL_{n+1}, we see that our space W_0 corresponds to the space of symmetric matrices in $\mathfrak{sl}_{n+1}$ where the entries above the superdiagonal are zero, and the height-one entries are precisely those in the superdiagonal. An explicit section of B can be taken to lie in $\frac{1}{4}W_0(\mathbb{Z})$: namely, if n is odd, the matrix

$$B(b_1,\dots,b_{n+1}) = \begin{pmatrix} 0 & 1 & & & & & & & \\ & 0 & \ddots & & & & & & \\ & & & 1 & & & & & \\ & & & 0 & 1 & & & & \\ & & & \frac{-b_2}{2} & -b_1 & 1 & & & \\ & & \iddots & -b_3 & \frac{-b_2}{2} & 0 & 1 & & \\ & \frac{-b_{n-2}}{2} & \iddots & \iddots & & & & \ddots & \\ \frac{-b_n}{2} & -b_{n-1} & \frac{-b_{n-2}}{2} & & & & & 0 & 1 \\ -b_{n+1} & \frac{-b_n}{2} & & & & & & & 0 \end{pmatrix}$$

can be seen to have characteristic polynomial $f(x) = x^{n+1} + b_1x^n + \cdots + b_nx + b_{n+1}$. (if n is even, a similar matrix can be given). The main observation in this case is that if m^2 weakly divides $\Delta(f)$, then there exists an $l \in \mathbb{Z}$ such that $f(x+l) = x^{n+1} + p_1x^n + \cdots + mp_nx + m^2p_{n+1}$ (cf. [BSW22a, Proposition 2.2]). Then, if $D = \operatorname{diag}(m, 1, \dots, 1, m^{-1})$, we observe that the matrix

$$D(B(p_1,\dots,p_{n-1},mp_n,m^2p_{n+1}) + lI_{n+1})D^{-1}$$

is integral, has characteristic polynomial $f(x)$ and the entries in the superdiagonal are $(m, 1, \dots, 1, m)$. Thus, this matrix has Q-invariant m, as desired.

Remark 3.6. Our Q-invariant is slightly different to the Q-invariant defined in [BSW22a], which is defined in a slightly more general subspace of V. When restricting to $W_0(F)$, their Q-invariant turns out to be a product of powers of the elements of the superdiagonal, whereas in our case we simply take the product of these elements. This difference does not affect the proof of Theorem 1.2, and we can also see that for both definitions the Q-invariant in the previous example is m.

4 Reduction theory

Before we are able to prove our main results, we need to stablish some results about reduction theory. Mainly, we will be concerned about defining appropriate heights in the GIT quotient B, and constructing a suitable "box-shaped" fundamental domain for the action of Γ on $G(F_\infty)$, where Γ will be an arithmetic subgroup of $G(F)$. In the future, we will use these constructions for $\Gamma = G_{\beta_i}$, where G_{β_i} was defined in Section 3.

4.1 Heights

Recall that M_∞ is the set of archimedean places of F. As stated in the introduction, for an element $b \in B(F)$ we define its *height* to be

$$\operatorname{ht}(b) := (NI_b) \prod_{v \in M_\infty} \sup\left(|p_{d_1}(b)|_v^{1/d_1}, \dots, |p_{d_r}(b)|_v^{1/d_r} \right),$$

where I_b is the scaling ideal $\{a \in F \mid a^{d_i}p_{d_i}(b) \in \mathcal{O}_F, \forall i\}$. We can check that this height is $\mathbb{G}_m(F)$-invariant using the product formula. A consequence of this is that when $|M_\infty| > 1$, the set of elements of $B(\mathcal{O}_F)$ having height less than X is infinite. To remedy that, instead of counting elements in $B(\mathcal{O}_F)$ we will count the number of elements of $\mathbb{G}_m(F)\backslash B(F)$ having height less than X. We have the following result by Deng (see [Den98, Theorem (A)]):

Theorem 4.1. *We have*

$$\#\{b \in \mathbb{G}_m(F)\backslash B(F) \mid \mathrm{ht}(b) \le X\} = C_B X^{\sum_i d_i} + O(X^{\sum_i d_i - \delta_B}),$$

where C_B, δ_B are real positive constants.

The constants C_B and δ_B are given explicitly in the statement of [Den98, Theorem (A)].

Remark 4.2. Theorem 4.1 is the main reason why we are choosing to work with this height. There are other natural heights that might be considered, such as the Weil height:

$$\mathrm{ht}_{\mathrm{Weil}}(b) = \prod_{v \in M_F} \sup_i \{|p_{d_i}(b_v)|_v^{1/d_i}\},$$

where now the product is taken over all places of F, finite and infinite. This product is also $\mathbb{G}_m(F)$-invariant by the product formula; however, we are not aware of any asymptotics for this height in the style of Theorem 4.1. Having results like that will be very useful when performing the geometry-of-numbers arguments in Section 5.

Moreover, there is a natural intepretation for our choice of height. In [ESZ23], a natural height on stacks is defined, which in the particular case of weighted projective spaces turns out to agree with our choice of height (cf. [ESZ23, §3.3]).

We note the following fact about the quantity $\sum_i d_i$ (see [Lag22a, Lemma 8.1]):

Proposition 4.3. *We have $\sum_i d_i = \dim V$.*

For our argument, it will be useful to construct a set $\Sigma \subset B(\mathcal{O}_F)$ which is a fundamental domain for the action of $\mathbb{G}_m(F)$ over $B(F)$, so that it suffices to count elements with invariants in Σ to prove Theorem 1.2. We will do so following [BSW, §3.4]. For a finite prime $\mathfrak{p}$, we set

$$B'(\mathcal{O}_\mathfrak{p}) = \{b \in B(\mathcal{O}_\mathfrak{p}) \mid v_\mathfrak{p}(p_{d_i}(b)) < d_i \text{ for some } i\}.$$

For every $b_\mathfrak{p} \in B(F_\mathfrak{p})$, there exists $g_\mathfrak{p} \in \mathbb{G}_m(F_\mathfrak{p})$ such that $g_\mathfrak{p} b_\mathfrak{p} \in B'(\mathcal{O}_\mathfrak{p})$; and moreover this $g_\mathfrak{p}$ is unique up to the action of $\mathbb{G}_m(\mathcal{O}_\mathfrak{p})$. In a similar spirit to Section 3, for any $\gamma \in \mathrm{cl}(\mathbb{G}_m)$ we can consider the sets

$$B_\gamma = B(F) \cap \gamma^{-1} \prod_\mathfrak{p} B'(\mathcal{O}_\mathfrak{p});$$
$$\mathbb{G}_{m,\gamma} = \mathbb{G}_m(F) \cap \gamma^{-1} \prod_\mathfrak{p} \mathbb{G}_m(\mathcal{O}_\mathfrak{p})\gamma = \mathcal{O}_F^\times.$$

Let $\gamma_1, \ldots, \gamma_k$ be representatives of $\mathrm{cl}(\mathbb{G}_m)$, which is finite by Proposition 2.10 (in fact, it coincides with the class group of F). Then, we have a bijection

$$\coprod_{i=1}^k \mathcal{O}_F^\times \backslash B_{\gamma_i} \longleftrightarrow \mathbb{G}_m(F)\backslash B(F),$$

given by inclusion. Indeed, if for some $g \in \mathbb{G}_m(F)$ we have $v_1 \in B_\gamma$ and $gv \in B_{\gamma'}$, then for all primes $\mathfrak{p}$ we have that $\gamma_\mathfrak{p} v \in B'(\mathcal{O}_\mathfrak{p})$ and $\gamma'_\mathfrak{p} gv \in B'(\mathcal{O}_\mathfrak{p})$. This implies that $\gamma'_\mathfrak{p} g \gamma_\mathfrak{p}^{-1} \in \mathbb{G}_m(\mathcal{O}_\mathfrak{p})$ which means that $\gamma = \gamma'$, from which injectiveness follows. The map is surjective by construction.

We can modify our choices of representatives γ_i of $\mathrm{cl}(\mathbb{G}_m)$ so that $B_{\gamma_i} \subset B(\mathcal{O}_F)$, simply by multiplying by appropriate elements of $\mathbb{G}_m(F)$. Moreover, for a given choice of γ_i, the ideal I_b is independent of the choice of $b \in B_{\gamma_i}$: in fact, a computation shows that I_b is equal to the ideal corresponding to γ_i, regardless of the choice of b (as long as $b \neq 0$).

To construct the fundamental domain Σ, it suffices to construct fundamental domains Σ_i in each B_{γ_i} separately. Given that in the future we will want to impose congruence conditions in Σ_i, we will also define Σ_i itself as a set defined by congruence conditions: that is, defined as the intersection of sets $\Sigma_{i,\mathfrak{p}} \subset B(\mathcal{O}_\mathfrak{p})$ for all primes $\mathfrak{p}$ (finite and infinite).

For finite primes $\mathfrak{p}$, the subset $\Sigma_{i,\mathfrak{p}}$ is given by $\gamma_\mathfrak{p}^{-1}B'(\mathcal{O}_\mathfrak{p})$. For infinite primes, we consider the set $F_\infty = \prod_{v\in F_\infty} F_v$ and the subset F^1_∞ of F_∞ consisting of those elements $(\alpha_v)_{v\in M_\infty}$ such that $\prod_v |\alpha_v|_v = 1$. We further consider Λ, a compact subset of F^1_∞ such that $F^1_\infty = \Lambda\mathcal{O}_F^\times$. Let $B(1)$ be the set of elements $b \in B(F_\infty)$ such that, for all $v \in M_\infty$:

$$\sup_i\{|p_{d_i}(b)|_v^{1/d_i}\} = \mathrm{ht}(b)^{1/|M_\infty|}.$$

Let $\overline{B}(1)$ be a measurable set with boundary of measure 0 which is a fundamental domain for the action of the roots of unity μ_F over $B(1)$. Then, the set $\Lambda\cdot(\overline{B}(1))$ is a fundamental domain for the action on $\mathcal{O}^\times$ over $B(F_\infty)$, which we take as our $\Sigma_{i,\infty}$. In conclusion, we obtain our fundamental domain Σ_i by combining the congruence conditions $\Sigma_{i,\mathfrak{p}}$ and $\Sigma_{i,\infty}$.

The fact that $\Sigma \cap B(F)_X$ is finite follows from the fact that Σ_∞ consists of elements whose local heights differ at most by an absolute constant. Thus, if an element has bounded height, then each of the local heights has to be bounded, yielding a finite number of elements overall.

4.2 Measures on G

Recall that $\Phi_G = \Phi(G,T)$ is the set of roots of G. The Borel subgroup P^+ of G determines a root basis S_G and a set of positive/negative roots $\Phi_G^\pm$, compatible with the choice of positive roots in H determined by the pinning of Section 2.1. Let N be the unipotent radical of the negative Borel subgroup P. We will make an appropriate choice of maximal compact subgroup of $G(F_\infty)$:

Lemma 4.4. *There exists a maximal compact subgroup K of $G(F_\infty)$ such that $T(F_\infty)\cap K = \{t = (t_1,\dots,t_r) \in T(F_\infty) \mid |t_i|_v = 1, \forall v \in M_\infty, \forall i\}$.*

Proof. We will choose a maximal compact $K_v \subset G(F_v)$ for every infinite place v satisfying that $T(F_v)\cap K_v = \{t = (t_1,\dots,t_r) \in T(F_v) \mid |t_i|_v = 1, \forall i\}$. Then, it will be enough to define $K = \prod_v K_v$.

Assume that $F_v = \mathbb{R}$. By the Isomorphism Theorem (see e.g. [Con14, Theorem 6.1.17]), the involution corresponding to -1 in the root system Φ_G induces an involution $\vartheta\colon G(F_v) \to G(F_v)$ that acts as inversion in $T(F_v)$. Moreover, this involution ϑ is a Cartan involution: to check this, we need to verify that the form $B_\vartheta(X,Y) := -B(X,\vartheta(Y))$ is positive definite (here, B is the Killing form). If we fix basis elements $X_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \neq 0$ in Φ_G, we can additionally require that $d\vartheta(X_\alpha) = -X_{-\alpha}$. Then, it is straightforward to check that B_ϑ is positive definite.

Now, assume that $F_v = \mathbb{C}$. An involution $\vartheta\colon G \to G$ is a Cartan involution if and only if $\{g \in G(\mathbb{C}) \mid \vartheta(g) = \overline{g}\}$ is a maximal compact subgroup of $G(\mathbb{C})$. Note that in the split torus T, the involution $\vartheta_T\colon T \to T$ sending $t \in T(\mathbb{C})$ to t^{-1} is a Cartan involution: if $t = (t_1,\dots,t_k) \in (\mathbb{C}^\times)^k$, then $\vartheta_T(t) = \bar{t}$ if and only if $|t_i| = 1$ for all i, so the set $\{t \in T(\mathbb{C}) \mid \vartheta_T(t) = \bar{t}\}$ is a maximal compact subgroup of $T(\mathbb{C})$. By [AT18, Theorem 3.13(1)(b)], the Cartan involution ϑ_T extends to a Cartan involution $\vartheta\colon G \to G$. By taking $K_v = \{g \in G(\mathbb{C}) \mid \vartheta(g) = \overline{g}\}$, we see that $T(F_v)\cap K_v$ corresponds exactly to those elements with modulus 1, as wanted. □

Consider the subgroup $A \subset T(F_\infty)$ of elements $a = (a_v)_{v\in M_\infty}$ such that $a_v \in \mathbb{R}_{>0}$ for all $v \in M_\infty$. Then, the map

$$N(F_\infty)\times A\times K \to G(F_\infty)$$

given by $(n,t,k) \mapsto ntk$ is a diffeomorphism. The following result is a well-known property of the Iwasawa decomposition:

Lemma 4.5. *Let dn, dt, dk be Haar measures on $N(F_\infty), A, K$, respectively. Then, the assignment*

$$f \mapsto \int_{n\in N(F_\infty)} \int_{t\in A} \int_{k\in K} f(ntk)|\delta(t)|^{-1} dn\, dt\, dk$$

defines a Haar measure on $G(F_\infty)$. Here, $\delta(t) = \prod_{\beta\in\Phi_G^-} \beta(t) = \det \mathrm{Ad}(t)|_{\mathrm{Lie}\,\overline{N}(F_\infty)}$ is an algebraic character obtained from the action of T on N, and $|\delta(t)| = \prod_{v\in M_\infty} |\delta(t_v)|_v$.

We get the Haar measure dt from the isomorphism with $\prod_{v\in M_\infty}(\mathbb{R}_{>0})^{\#S_G}$, where $\mathbb{R}_{>0}$ is given the standard Haar measure $d^\times\lambda = d\lambda/\lambda$.

4.3 Fundamental sets

In this section, given an arithmetic subgroup $\Gamma \subset G(F)$, we will construct an exact fundamental domain $\mathcal{F}$ for the action of Γ on $G(F_\infty)$.

Definition 4.6. A *Siegel set* is a set of the form $\mathcal{S} = \cup_{i=1}^n \alpha_i\omega_i A_c K$, where $\alpha_i \in G(F)$, the sets $\omega_i \subset N(F_\infty)$ are compact and

$$T_c = \{t \in T(F_\infty) \mid |\alpha(t)| \le c,\ \forall \alpha \in S_G\}, \quad A_c = T_c \cap A.$$

Remark 4.7. In fact, we will consider subsets of the form $\mathcal{S} = \cup_{i=1}^n \alpha_i\omega_i A'_c K_1$, where A'_c and K_1 are appropriate subsets of A_c and K, respectively. We will still call such a set a Siegel set, for simplicity.

We will require our fundamental domain $\mathcal{F}$ to be "box-shaped at infinity", in the following sense:

Definition 4.8. We say a fundamental domain $\mathcal{F}$ for the action of Γ on $G(F_\infty)$ is *box-shaped at infinity* if there exist Siegel sets $\mathcal{S}_1 \subset \mathcal{F} \subset \mathcal{S}_2$ such that

- $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same cusps, i.e. both sets are defined as $\mathcal{S}_1 = \cup_{i=1}^n \alpha_i\omega_i A'_{c_1} K_1$ and $\mathcal{S}_2 = \cup_{i=1}^n \alpha_i\omega_i A'_{c_2} K_1$ for the same choice of elements $\alpha_i \in G(F)$ in each case, and the same choice of subsets A'_c and K_1.
- There exists an open subset $\mathcal{U}_1 \subset \mathcal{S}_1$ of full measure such that each Γ-orbit on $G(F_\infty)$ intersects $\mathcal{U}_1$ at most once.
- Every Γ-orbit on $G(F_\infty)$ intersects $\mathcal{S}_2$ at least once.
- For sufficiently small c, we have that $\mathcal{S}_1 \cap (\cup_{i=1}^n \alpha_i N(F_\infty)A_c K) = \mathcal{S}_2 \cap (\cup_{i=1}^n \alpha_i N(F_\infty)A_c K)$.

In what follows, it will be sufficient to construct $\mathcal{S}_1$ and $\mathcal{S}_2$ to obtain $\mathcal{F}$ due to the following lemma (cf. [Sha+22, Lemma 7]):

Lemma 4.9. *Let $B(G)$ be the Borel σ-algebra of $G(F_\infty)$. Assume we have $\mathcal{S}_1$ and $\mathcal{S}_2$ in $B(G)$ such that the maps $\mathcal{S}_1 \to \Gamma\backslash G(F_\infty)$ and $\mathcal{S}_2 \to \Gamma\backslash G(F_\infty)$ are injective and surjective, respectively. Then, there is a fundamental domain $\mathcal{F}$ in $B(G)$ for the action of Γ in $G(F_\infty)$ such that $\mathcal{S}_1 \subset \mathcal{F} \subset \mathcal{S}_2$.*

Proof. Since Γ is a discrete subgroup of $G(F_\infty)$, we can find a non-empty open set $U \subset G(F_\infty)$ such that $U^{-1}U \cap \Gamma = \{\mathrm{id}\}$. Given that $G(F_\infty)$ is second countable, we can find countably many elements $\{g_n\}_{n\in\mathbb{N}} \subset G(F_\infty)$ such that $G(F_\infty) = \cup_{n=1}^\infty g_n U$. Let $\mathcal{S}_3 = \mathcal{S}_2 \setminus \Gamma\mathcal{S}_1$, and define:

$$\mathcal{F}_0 = \bigcup_{n=1}^\infty \left(g_n U \cap \mathcal{S}_3 \setminus \bigcup_{i<n} \Gamma(g_i U \cap \mathcal{S}_3) \right).$$

Finally, set $\mathcal{F} = \mathcal{S}_1 \cup \mathcal{F}_0$, a disjoint union. Then, $\mathcal{F} \in B(G)$ and the map $\mathcal{F} \to \Gamma\backslash G(F_\infty)$ can easily be seen to be bijective, as wanted. □

4.3.1 Constructing $\mathcal{S}_1$

To obtain the domain $\mathcal{S}_1$, we will use general properties of the Borel-Serre compactification following [BS73].

First, consider the Weil restriction of scalars $G' = \operatorname{Res}_{F/\mathbb{Q}} G$, which is a semisimple group defined over $\mathbb{Q}$. We have an isomorphism $\phi_G\colon G'(\mathbb{Q}) \cong G(F)$, inducing $\phi_G\colon G'(\mathbb{R}) \cong G(F_\infty)$. Even though the group G is split with maximal torus T, the group G' will *never* be split (unless $F = \mathbb{Q}$). Let us denote $T' = \operatorname{Res}_{F/\mathbb{Q}} T$, a maximal torus of G' and $T'_{split} \subset T'$ the maximal $\mathbb{Q}$-split torus inside T'.

Recall that the F-split torus T is obtained by base-changing a $\mathbb{Q}$-split torus $T_{\mathbb{Q}}$ to F. In particular, there is an isomorphism over $\mathbb{Q}$ between the split tori $T'_{split} \cong T_{\mathbb{Q}}$, inducing an isomorphism of character groups $X^*(T'_{split}) \cong X^*(T_{\mathbb{Q}})$. By [BT65, (6.21)] this isomorphism induces a bijection between the root systems $\Phi(G,T)$ and $\Phi(G', T'_{split})$. Denote by $S_{G'}$ a positive root basis of $\Phi(G', T'_{split})$, chose compatibly with the positive root basis of $\Phi(G,T)$. Let us define

$$T'_c = \{t \in T'_{split}(\mathbb{R}) \mid |\beta(t)| \leq c,\ \forall \beta \in S_{G'}\}.$$

Let $P' = \operatorname{Res}_{F/\mathbb{Q}} P$ be a minimal parabolic subgroup of G' containing T', and define

$${}^0P'(\mathbb{R}) = \bigcap_{\chi \in X^*(P')} \ker \chi^2.$$

We then have that $P'(\mathbb{R}) = {}^0P'(\mathbb{R}) \rtimes A_{split}$ by [BS73, Proposition 1.2], where A_{split} denotes the connected component of $T'_{split}(\mathbb{R})$.

Proposition 4.10. *Let $c > 0$. Then, $\phi_G(T_c) \subset {}^0P'(\mathbb{R}) \rtimes (T'_{c'} \cap A_{split})$ for some positive constant c'. Similarly, $\phi_G^{-1}(T'_c) \subset T_{c'}$ for some c'.*

Proof. Let us begin by recalling the isomorphism $T_{\mathbb{Q}} \cong T'_{split}$. When base-changing to $\mathbb{R}$, we see that $\phi_G^{-1}(T'_{split}(\mathbb{R}))$ corresponds to the $t = (t_1, \ldots, t_k) \in T(F_\infty)$ such that each $t_i = (t_{i,v})_{v \in M_\infty} \in F_\infty$ satisfies $t_{i,v} = t_{i,v'} \in \mathbb{R}^\times$ for all infinite places v, v'.

Let $t \in T_c$. We can decompose $t = t_1 t_2$ with $t_1 \in \phi_G^{-1}(T'_{split}(\mathbb{R}))$, scaled in such a way that $\prod_v |\alpha(t_v)|_v = \prod_v |\alpha(t_{1,v})|_v$ for all $\alpha \in X^*(T)$. By [BT65, (6.20)], there is an isomorphism $X^*(T)_F \xrightarrow{F} X^*(T')_{\mathbb{Q}}$ such that for all $\chi \in X^*(T)$ and $g \in G(F_\infty)$, we have that $\prod_v |\chi(g_v)|_v = |(F \circ \chi)(\phi_G(g))|$. In particular, we have that $\phi(t_2) \in {}^0P'(\mathbb{R})$. Finally, for any $t_2 \in \phi_G^{-1}(T'_{split}(\mathbb{R}))$, we have that $t_2 \in T_c$ if and only if $\phi_G(t_2) \in T'_{c^{1/[F:\mathbb{Q}]}}$, so choosing $c' = c^{1/[F:\mathbb{Q}]}$ concludes the proof of the first inclusion. The second inclusion is analogous. □

We will denote by K' the restriction of the maximal compact K inside G'. Now, consider the symmetric space $X = G'(\mathbb{Q})/K'$. For each parabolic $\mathbb{Q}$-subgroup Q of G', let $S_Q := (R_d Q/(R_u Q \cdot R_d G'))$, where R_u denotes the unipotent radical and R_d denotes the $\mathbb{Q}$-split part. Then, S_Q is a $\mathbb{Q}$-split torus, and we let $A_Q := S_Q(\mathbb{R})^\circ$. There is a natural action of A_Q on X; called the geodesic action (see [BS73, (3.2)]). Set $e(Q) = A_Q \backslash X$, and consider

$$\overline{X} = \coprod_{P \subset G' \text{ parabolic}} e(Q),$$

which by [BS73, (7.1)] naturally has a structure of a manifold with corners. The topology of $\overline{X}$ is studied in [BS73, §5, §6]; in particular, it is shown that for any parabolic group Q, the subset $X(Q) = \coprod_{R \supset Q} e(R)$ is an open subset of $\overline{X}$. Taking $Q = G$, we see that $e(G) = X$ is an open submanifold of $\overline{X}$.

Now, let us return to considering the minimal parabolic $P' = \mathrm{Res}_{K/\mathbb{Q}} P$, and consider

$$U_{x,P',c} = {}^0P'(\mathbb{R})(A_{P',c} \cdot x),$$

where $x \in X$ and $A_{P',c} = A_{split} \cap T'_c$, as defined in the beginning of the section. The closure $\overline{U_{x,P',c}}$ in $\overline{X}$ is a neighbourhood of the closure of $e(P')$ in $\overline{X}$. The arithmetic subgroup Γ of $G(F)$ restricts to an arithmetic subgroup Γ' of $G'(\mathbb{Q})$: when we consider the action of Γ' in these sets $U_{x,P',c}$, it is useful to consider the following result (see [BS73, Proposition 10.3]):

Proposition 4.11. *There exists $c > 0$ satisfying that for any $g_1, g_2 \in \overline{U_{x,P',c}}$, if there exists $\gamma \in \Gamma'$ such that $g_1 = \gamma g_2$, then $\gamma \in P' \cap \Gamma'$.*

For our construction of $\mathcal{S}_1$, we will have to worry about different cusps at once. Let $\{\alpha_1, \ldots, \alpha_m\}$ be a set of representatives for the double cosets of $\Gamma\backslash G(F)/P(F)$, which is equivalent to choosing a set of representatives $\{\alpha'_1, \ldots, \alpha'_m\}$ for the double cosets of $\Gamma'\backslash G'(\mathbb{Q})/P'(\mathbb{Q})$. The interaction of different cusps can be controlled as follows (see [BS73, Proposition 10.4]):

Proposition 4.12. *Let Q, R be parabolic subgroups of G', let $x, y \in X$ and let $g \in G'(\mathbb{Q})$. Then, $g \cdot U_{x,Q,c} \cap U_{y,R,c} \neq \emptyset$ for all $c > 0$ if and only if $gQg^{-1} \cap R$ is parabolic.*

In particular, when Q is a minimal parabolic subgroup and $Q = R$, we have that $gQg^{-1} \cap Q$ is parabolic exactly when $g \in Q$.

We will take our set $\mathcal{S}_1$ in $G(F_\infty)$ to be contained in a set of the form $\cup_{i=1}^m \alpha_i \omega A_c K$, where $\alpha_i \in G(F)$ are as above. We will choose c to be "big enough":

Proposition 4.13. *There exists a small enough $c > 0$ such that if $g_1 \in \alpha_i N(F_\infty) A_c K$ and $g_2 \in \alpha_j N(F_\infty) A_c K$ are Γ-equivalent, then $\alpha_i = \alpha_j$.*

Proof. By Proposition 4.10, we have that $\phi_G(N(F_\infty)A_cK) \subset {}^0P'(\mathbb{R})T'_{c'}K$ for some $c' > 0$. We choose $c > 0$ small enough so that c' satisfies the conclusions of Proposition 4.11. Then, if there exists $\gamma \in \Gamma$ such that $\gamma g_1 = g_2$, then we would have that $\alpha_j^{-1}\gamma\alpha_i \in P(F)$, or in other words that $\alpha_i \in \Gamma\alpha_j P(F)$, which can only happen if $\alpha_i = \alpha_j$. □

Therefore, we can choose c small enough so that there are no intersections between the cusps. Now, we need to choose appropriate subsets of $N(F_\infty)$, A_c and K so that each cusp contains at most one Γ-orbit.

Lemma 4.14. *There exist compact sets ω_i inside $N(F_\infty)$ such that every $\alpha_i^{-1}\Gamma\alpha_i \cap N$-orbit intersects ω_i, and also satisfying that inside a set of full measure, every $\alpha_i^{-1}\Gamma\alpha_i \cap N$-orbit intersects ω_i exactly once.*

Proof. First, we recall that for any non-zero root $\beta \in \Phi_G$, there is an isomorphism $u_\beta \colon \mathbb{G}_a \to U_\beta$, for some subgroup $U_\beta \subset G$. Let $\beta_1, \ldots, \beta_m$ be an ordering of the roots of Φ_G^-, such that $|\operatorname{ht}(\beta_i)| \leq |\operatorname{ht}(\beta_{i+1})|$. Then, by [Con14, Theorem 5.1.13], there is an isomorphism of algebraic varieties (not necessarily a group morphism)

$$\prod_{i=1}^m U_{\beta_i} \to N$$

defined over $\mathbb{Z}$, which is just given by the product map. Given that $N_i := \alpha_i^{-1}\Gamma\alpha_i \cap N$ is an arithmetic subgroup of $N(F)$, we can arrange everything so that the elements of N_i correspond precisely to those elements $u_{\beta_1}(x_1)\ldots u_{\beta_m}(x_m)$ such that $x_m \in \mathcal{O}_F$. Choose a compact subset Λ of F_∞ such that $\mathcal{O}_F + \Lambda = F_\infty$, such that the interior of Λ has full measure inside Λ, and such that no two elements in the interior differ by an element of $\mathcal{O}_F$. Now, set

$$\omega_i = \{u_{\beta_1}(x_1)\cdots u_{\beta_m}(x_m) \mid x_1, \ldots, x_m \in \Lambda\}.$$

Using the commutator relations in [Con14, Proposition 5.1.14], it is not difficult to see that ω_i satisfies the conclusions of the lemma. □

Now, we worry about the action of T on A_c and K. More specifically, we need to account for the action of the group $T_i := \alpha_i^{-1}\Gamma\alpha_i \cap T(F)$, which is an arithmetic subgroup of $T(F)$. In particular, it has to be commensurable to $T(\mathcal{O}_F)$, which means in particular that $T_i \subset T(\mathcal{O}_F)$.

Let $w_F \subset \mathcal{O}_F^*$ be the subgroup of roots of unity of $\mathcal{O}_F$. Under the identification $T(\mathcal{O}_F) \cong (\mathcal{O}_F^*)^{\# S_G}$, consider the subgroup $T_{w,i} \subset T_i$ corresponding to the appropriate subgroup of $(w_F)^{\# S_G}$ lying in T_i. Then, we know that $T_{w,i} \subset K$ by Lemma 4.4. We consider a fundamental domain for the left action of $T_{w,i}$ on K, which we will denote K_1.

Let $|T_i|$ denote the subset of $A(F_\infty)$ which is the image of T_i under the projection map that sends $t = (t_1, \dots, t_k) \in T_i$ to $(a_1, \dots, a_k)$, where $a_{j,v} = |t_j|_v$. We denote a fundamental domain for the action of $|T_i|$ on A_c by A'_c.

Finally, we let $\mathcal{S}_1 = \cup_{i=1}^m \alpha_i\omega_i A'_c K_1$.

Theorem 4.15. *There exists an open subset $\mathcal{U}_1$ of $\mathcal{S}_1$ of full measure such that any Γ-orbit in $G(F_\infty)$ intersects $\mathcal{U}_1$ at most once.*

Proof. Let $\omega'_i \subset \omega_i$ be the subset of full measure described in Lemma 4.14, and let

$$\mathcal{U}_1 = \cup_{i=1}^m \alpha_i\omega'_i A'_c K_1.$$

Now, let $g_1, g_2 \in \mathcal{U}_1$, and let $\gamma \in \Gamma$ be an element such that $\gamma \cdot g_1 = g_2$. By Proposition 4.12, we know that g_1 and g_2 have to lie in the same cusp; that is, we have $g_1, g_2 \in \alpha_i\omega'_i A'_c K_1$. Write $g_1 = \alpha_i n_1 t_1 k_1$ and $g_2 = \alpha_i n_2 t_2 k_2$. We have that by Proposition 4.11 that $\alpha_i^{-1}\gamma\alpha_i \in P$, or in other words that $\gamma \in \alpha_i P \alpha_i^{-1} \cap \Gamma$. Let $\gamma = \alpha_i n_0 t_0 \alpha_i^{-1}$, where $n_0 \in \alpha_i^{-1}\Gamma\alpha_i \cap N(F)$ and $t_0 \in \alpha_i^{-1}\Gamma\alpha_i \cap T(F)$. Then, the condition that $\gamma \cdot g_1 = g_2$ becomes

$$\alpha_i n_0 (t_0 n_1 t_0^{-1}) t_0 t_1 k_1 = \alpha_i n_2 t_2 k_2.$$

Let us write $t_0 = t_a t_k$, where $t_a \in A(F_\infty)$ and $t_k \in K$. Then, the uniqueness in the Iwasawa decomposition gives us $n_0 t_0 n_1 t_0^{-1} = n_2$, $t_a t_1 = t_2$ and $t_k k_1 = k_2$. By construction of A'_c, we must have that $t_a = 1$, and therefore that $t_k \in T_w$, so by construction of K_1 we also have that $t_k = 1$. Then $t_0 = 1$, so the equation $n_0 n_1 = n_2$ also gives $n_0 = 1$ by construction of ω'_i. ☐

4.3.2 Constructing $\mathcal{S}_2$

Having chosen $\mathcal{S}_1$, we will now construct a compatible $\mathcal{S}_2$.

Proposition 4.16. *There exists $c > 0$ such that $G(F_\infty) = \Gamma \cdot \cup_{i=1}^m \alpha_i\omega_i A'_c K_1$, where α_i, ω_i, A'_c and K_1 are as in Section 4.3.1.*

Proof. Using Proposition 4.10 and restriction of scalars, the results in [Spr94] tell us that there exists a $c > 0$ such that

$$G(F_\infty) = \Gamma \cdot \bigcup_{i=1}^m \alpha_i \omega' T_c K$$

for some compact subset $\omega' \in N(F_\infty)$. Now, assume that we have $g \in G(F_\infty)$ written as $g = \alpha_i n_0 t_0 k_0$ for some $n_0 \in \omega'$, $t_0 \in T_c$ and $k_0 \in K$. We want to see that $g \in \Gamma \cdot \cup_{i=1}^m \alpha_i\omega_i A'_c K_1$.

By construction, we will have that $t_0 k_0 = t_\Gamma t_1 k_1$, where $t_\Gamma \in \alpha_i^{-1}\Gamma\alpha_i \cap T(F)$, $t_1 \in A'_c$ and $k_1 \in K_1$. We also have that there exists $n_\Gamma \in \alpha_i^{-1}\Gamma\alpha_i \cap N(F)$ and $n_1 \in \omega'_i$ such that $n_\Gamma t_\Gamma n_0 t_\Gamma^{-1} = n_1$. Then, we have that

$$(\alpha_i n_\gamma t_\Gamma \alpha_i^{-1}) \cdot \alpha_i n_0 t_0 k_0 = \alpha_i n_\gamma t_\Gamma n_0 t_\Gamma^{-1} t_\Gamma t_1 k_1 = \alpha_i n_1 t_1 k_1,$$

as wanted. ☐

We choose $\mathcal{S}_2$ to be the set constructed in Proposition 4.16. It is clear that $\mathcal{S}_1$ and $\mathcal{S}_2$ satisfy the conditions stated in Definition 4.8. Therefore, by Lemma 4.9, we have constructed the desired box-shaped fundamental domain $\mathcal{F}$.

For the future, it will be useful to record the following property of $\mathcal{S}_2$ (known as the Siegel property):

Proposition 4.17. *The size of the fibres of the map $\mathcal{S}_2 \to \Gamma\backslash G(F_\infty)$ is uniformly bounded.*

Proof. It suffices to show that the set $\{\gamma \in \Gamma \mid \gamma\mathcal{S}_2 \cap \mathcal{S}_2 \neq \emptyset\}$ is finite. For $F = \mathbb{Q}$, this follows from [Bor69, Corollaire 15.3], and in the general case we reduce to $F = \mathbb{Q}$ using restriction of scalars. □

5 Counting reducible orbits

Before we are able to prove Theorem 1.2, we need to obtain an estimate on the number of reducible Γ-orbits on some Γ-invariant lattice V_0 of $V(F_\infty)$ which is commensurable with $V(\mathcal{O}_F)$. Given that, as observed in Section 4.1, there might be infinitely many orbits with bounded height, we will restrict to the elements with invariants lying on the fundamental domain $\Sigma \subset B(\mathcal{O}_K)$. Recall that Σ was the disjoint union of finitely many sets Σ_i, where for each $b \in \Sigma_i$ we had

$$\mathrm{ht}_i(b) = C_i \prod_{v \in M_\infty} \sup\{|p_{d_i}(b)|_v^{1/d_i}\}, \tag{3}$$

for some constant C_i only dependent on i (not on the choice of b inside Σ_i). We will count the number of reducible Γ-orbits in V_0 having invariants in Σ_i for each i separately. For simplicity, we will denote Σ_i simply by Σ in this section.

For any element $b \in B(F_\infty)$, we *define* its height to be given by the expression in (3), and for any $v \in V(F_\infty)$ we also define $\mathrm{ht}(v) := \mathrm{ht}_i(\pi(v))$.

Let Λ be the embedding of $\mathbb{R}_{>0}$ inside F_∞ given by sending $x \in \mathbb{R}_{>0}$ to $|M_\infty|$ copies of x inside every infinite place of F. For an element $\lambda \in \Lambda$ and $v \in V(F_\infty)$, we have that $\mathrm{ht}(\lambda v) = \lambda^{[F:\mathbb{Q}]}\,\mathrm{ht}(v)$, or in other words that the function ht is homogeneous of weight $[F:\mathbb{Q}]$.

We will prove the following:

Theorem 5.1. *Let $N(\Gamma, V_0, X)$ denote the number of reducible Γ-orbits in V_0 having height less than X and invariants lying in $\Lambda\Sigma$. Then, we have*

$$N(\Gamma, V_0, X) = CX^{\dim V} + O(X^{\dim V - \delta}).$$

The constant C depends only on Γ and V_0, and the constant δ can be chosen independently of Γ and V_0.

5.1 Averaging and reductions

By analogous arguments to [Tho15, §2.9], there exist open subsets $L_1, \ldots, L_k$ covering $\{b \in B(F_\infty) \mid \mathrm{ht}(b) = 1,\, \Delta(b) \neq 0\}$ such that for a fixed i, the quantity $r_i := \#\,\mathrm{Stab}_G(v)(F_\infty)$ remains constant for any choice of $v \in \pi^{-1}(L_i)$. Let us denote by Λ the embedding of $\mathbb{R}_{>0}$ inside F_∞ that sends $x \in \mathbb{R}_{>0}$ to $(x, \ldots, x) \in F_\infty$, and denote $V_i := V_0^{red} \cap G(F_\infty)\Lambda\kappa(L_i)$. Fix a compact left and right K-invariant set $G_0 \subset G(\mathbb{R})$ which is the closure of a non-empty open set, for which we assume that $G_0 = G_0^{-1}$. An averaging argument just as in [BS15, §2.3]) yields

$$N(\Gamma, V_i, X) = \frac{1}{r_i \,\mathrm{vol}(G_0)} \int_{g \in \mathcal{F}} \#\{v \in V_0^{red} \cap (gG_0\Lambda\kappa(L_i \cap \Sigma))_{<X}\}dg. \tag{4}$$

Here, the subscript $< X$ means we are restricting to elements of height less than X. For simplicity, we will denote $\mathcal{B}_X = (G_0\Lambda\kappa(L_i \cap \Sigma))_{<X}$.

In what follows, we will use the following version of Davenport's lemma [Dav51].

Proposition 5.2. *Let $\mathcal{R}$ be a bounded, semialgebraic multiset in $\mathbb{R}^n$ having maximum multiplicity m and that is defined by at most k polynomial inequalities, each having degree at most l. Let L be a rank n lattice inside $\mathbb{R}^n$. Then,*

$$\#(\mathcal{R} \cap L) = \mathrm{vol}_L(\mathcal{R}) + O(\max(\{\mathrm{vol}(\overline{\mathcal{R}}), 1\})),$$

Here, vol_L is a constant multiple of vol with $\mathrm{vol}_L(\mathbb{R}^n/L) = 1$, and $\mathrm{vol}(\overline{\mathcal{R}})$ denotes the greatest d-dimensional volume of any projection of $\mathcal{R}$ onto a coordinate subspace obtained by equating $n-d$ coordinates to zero, and where d takes any value between 1 and $n-1$. The implied constant in the second summand depends only on n, m, k and l.

We now want to prove a "cutting-off-the-cusp" result in the style of e.g. [Lag22a, Proposition 8.11], which should say that most elements "in the cusp" fall into the subspace W_0 of V defined in Section 3. However, unlike previous instances of this result, in our case we have multiple cusps to worry about. Given that $\mathcal{S}_1 \subset \mathcal{F} \subset \mathcal{S}_2$, we can write $\mathcal{F} = \cup_{j=1}^m \alpha_j \mathcal{S}_j$, where the $\alpha_j \in G(F)$ are as in Section 4.3 and $\mathcal{S}_j$ are subsets of $\omega_j A'_c K_1$. We can further assume that $\alpha_i \mathcal{S}_i \cap \alpha_j \mathcal{S}_j = \emptyset$ for all $i \neq j$. Then, we can write

$$N(\Gamma, V_i, X) = \frac{1}{r_i \,\mathrm{vol}(G_0)} \sum_{j=1}^m \int_{g \in \alpha_j \mathcal{S}_j} \#\{v \in V_0^{red} \cap (g\mathcal{B}_X)\} dg$$

For each cusp corresponding to α_j, we can consider the weights of the action of $\alpha_j T \alpha_j^{-1}$ on V (the weight spaces will be of the form $\alpha \cdot V_\lambda$, where V_λ are the weight spaces for the action of T).

Proposition 5.3. *Let $v_{0,j}$ denote the highest weight of V under the action of $\alpha_j T \alpha_j^{-1}$. Then, there exists a constant $\delta > 0$ such that*

$$\int_{g \in \alpha_j \mathcal{S}_j} \#\{v \in (V_0 \setminus (\alpha_j \cdot W_0)) \cap g\mathcal{B}_X \mid v_{0,j} = 0\} dg = O(X^{\dim V - \delta})$$

Proof. This proof will follow the argument in [BSW, §3.2] and [Tho15, §3.3 and §5]. Recall that $\mathcal{S}_2$ is a finite cover of $\mathcal{F}$ of absolutely bounded degree by the Siegel property (i.e. Proposition 4.17). Hence, we can assume that

$$\mathcal{S}_j = \omega_j A'_c K_1,$$

following the notation in Section 4.3. Without loss of generality, we may assume that $\alpha_j = 1$, since the statement for the rest of the cusps is analogous. There exists a compact subset $\omega' \subset N(F_\infty)$ that contains the union of all $t^{-1}\omega t$ as t varies in A'_c. Therefore, we have

$$\int_{g \in \mathcal{S}_j} \#\{v \in (V_0 \setminus W_0) \cap g\mathcal{B}_X \mid v_0 = 0\} dg \ll \int_{t \in A'_c} \#\{v \in (V_0 \setminus W_0) \cap t\omega'\mathcal{B}_X \mid v_0 = 0\} |\delta^{-1}(t)| dt$$

Let Φ_V denote the characters of V under the action of T. For two disjoint subsets $M_0, M_1 \subset \Phi_V$, we define $S(M_0, M_1) = \{v \in V_0 \mid v_a = 0, \forall a \in M_0, v_b \neq 0, \forall b \in M_1\}$. We define $\mathcal{C}$ to be the collection of subsets $M_0 \subset \Phi_V$ such that if $a \in M_0$ and $b \geq a$, then $b \in M_0$. Additionally, given $M_0 \in \mathcal{C}$, we define $\lambda(M_0) = \{a \in \Phi_V \setminus M_0 \mid M_0 \cup \{a\} \in \mathcal{C}\}$. For $M_0 \in \mathcal{C}$, we refer to a pair $(M_0, \lambda(M_0))$ as a cusp datum.

Any element in $(V_0 \setminus W_0) \cap g\mathcal{B}_X$ falls inside one of the subsets $S(M_0, M_1)$ for some cusp datum (M_0, M_1). Therefore, it suffices to prove that

$$N(\Gamma, S(M_0, M_1), X) = O(X^{\dim V - \delta})$$

for every cusp datum such that $S(M_0, M_1) \nsubseteq W_0$.

Now, fix such a cusp datum (M_0, M_1). We note that if $S(M_0, M_1) \cap g\mathcal{B}_X$ is non-empty, then we must have $X|a(t)| \gg 1$ for every $a \in M_1$. We also note that $\prod_{a\in\Phi_V} |a(t)| = 1$ for all $t \in T(F_\infty)$. Given that V_0 is commensurable with $V(\mathcal{O}_K)$, there exists a constant C_0 such that for every $\chi \in U_0$ and $v \in V_0$, we have that either $v_\chi = 0$ or $|v_\chi| \geq C_0$. We define

$$V_{M_0,M_1} := \{v \in V(F_\infty) \mid v_a = 0, \forall a \in M_0;\ |v_b| \geq C_0, \forall b \in M_1\}.$$

Then, we have the following estimate:

$$\mathrm{vol}(g\mathcal{B}_X \cap V_{M_0,M_1}) = X^{\dim V - \#M_0} \prod_{a\in M_0} |a(t)|^{-1}.$$

We also define

$$A(M_0, M_1, X) := \{t \in T^\theta(F_\infty) \mid t \in A'_c;\ |a(t)| \gg X^{-1}, \forall a \in M_1\}.$$

Finally, recall that $\delta^{-1}(t) = \prod_{a\in\Phi_G^+} a(t)$. Using Proposition 5.2 and the above observations, we get

$$N(\Gamma, S(M_0, M_1), X) \ll X^{\dim V - \#M_0} \int_{t\in A(M_0,M_1,X)} \prod_{a\in\Phi_G^+} |a(t)| \prod_{a\in M_0} |a(t)|^{-1} dt.$$

Therefore, we have reduced our statement to showing that

$$\int_{t\in A(M_0,M_1,X)} \prod_{a\in\Phi_G^+} |a(t)| \prod_{a\in M_0} |a(t)|^{-1} dt = O(X^{\#M_0-\delta}).$$

Denote $w(t) = \prod_{a\in\Phi_G^+} |a(t)| \prod_{a\in M_0} |a(t)|^{-1}$. Using a trick, due to Bhargava (cf. [Bha10, Lemma 19]), let us consider a function $f\colon M_1 \to \mathbb{R}_{\geq 0}$. We have that $\prod_{a\in M_1} (X|a(t)|)^{f(a)} \gg 1$, and therefore that

$$\int_{t\in A(M_0,M_1,X)} w(t)dt \ll X^{\sum_{a\in M_1} f(a)} \int_{t\in A(M_0,M_1,X)} w(t) \prod_{a\in M_1} |a(t)|^{f(a)} dt. \quad (5)$$

Recall that any element $a \in X^*(T) \otimes \mathbb{Q}$ can be written uniquely as $a = \sum_{\alpha_i\in S_G} n_i(a)\alpha_i$ for some rational numbers $n_i(a)$. If we all the exponents of $w(t) \prod_{a\in M_1} |a(t)|^{f(a)}$ in terms of the basis α_i are positive, then the second integral in (5) is bounded independently of X, and therefore

$$\int_{t\in A(M_0,M_1,X)} w(t)dt \ll X^{\sum_{a\in M_1} f(a)}.$$

Therefore, the proposition is reduced to finding a function $f\colon M_1 \to \mathbb{R}_{\geq 0}$ satisfying:

- We have $\sum_{a\in M_1} f(a) < \#M_0$.
- For all i, we have $\sum_{\beta\in\Phi_G^+} n_i(\beta) - \sum_{a\in M_0} n_i(a) + \sum_{a\in M_1} f(a)n_i(a) > 0$.

Note that this last condition is independent of the base field F, so it is sufficient to prove the cutting-off-the-cusp results over $\mathbb{Q}$. This is the content of [Lag22a, Proposition 8.21]. □

Hence, if an element is in the cusp (i.e. the highest weight has coefficient zero), then it always falls in W_0 (and is therefore reducible) expect for negligibly many cases. We will now see that, analogously, almost all the elements in the main body are irreducible:

Proposition 5.4. *There exists a constant $\delta > 0$ such that*

$$\int_{g\in\mathcal{F}} \#\{v \in V_0^{red} \cap g\mathcal{B}_X \mid v_0 \neq 0\}dg = O(X^{\dim V - \delta}).$$

Proof. To prove this, we will use the Selberg sieve, in an analogous way to [ST14]. The general argument in [ST14], for $F = \mathbb{Q}$, is the following: assume that for any translate L of $mV(\mathbb{Z})$ we have that

$$N^*(L \cap V_i, X) = c_i m^{-A} X^B + O(m^{-C} X^{B-D}), \tag{6}$$

where A, B, C, D and c_i are positive constants, and N^* is some orbit-counting function. Let $S = \cap_{p \text{ prime}} S_p \subset V(\mathbb{Z})$ be a set defined by infinitely many congruence conditions modulo p, with each set S_p having density λ_p. Assume that λ_p tends to some constant $\lambda \in (0, 1)$ as p tends to infinity. Then, it is shown that

$$N^*(S \cap V_i, X) = O(X^{B-\delta}),$$

for some constant $\delta > 0$ which can be obtained explicitly depending on A, B, C, D. If instead of working over $\mathbb{Q}$ we work over a number field F, we can apply the same argument substituting our sets S_p for $S_\mathfrak{p}$ for $\mathfrak{p}$ a prime ideal of $\mathcal{O}_F$, and using the version of the Selberg sieve stated in [Rie58, Satz 1].

In our case, we set

$$N^*(S, X) = \int_{g \in \mathcal{F}} \#\{v \in S \cap g\mathcal{B}_X \mid v_0 \neq 0\} dg.$$

First, we need a power saving estimate for $N^*(L \cap V_i, X)$, where L is a translate of IV_0 for some ideal $I \subset \mathcal{O}_F$. This is done in Corollary A.2. We now let

$$S_\mathfrak{p} = \{v \in V(k_\mathfrak{p}) \mid \Delta(v) = 0 \text{ or } v \text{ is } k_\mathfrak{p}\text{-reducible}\}.$$

By [Lag22a, Theorem 7.16], any reducible element in V_0 falls into the set $S_\mathfrak{p}$ for all primes not dividing N_{bad}, and [Lag22a, Proof of Lemma 8.20] shows that the density of the sets $S_\mathfrak{p}$ tends to some constant $c \in (0, 1)$ as $N\mathfrak{p}$ tends to infinity. In conclusion, we can apply the Selberg sieve to obtain the desired power saving estimate. □

Finally, we argue that instead of integrating over the fundamental domain $\mathcal{F}$, it is enough to integrate over the smaller and more convenient set $\mathcal{S}_1$:

Lemma 5.5. *There is some constant $\delta > 0$ such that*

$$\int_{g \in \mathcal{F} \setminus \mathcal{S}_1} \#\{v \in V_0^{red} \cap g\mathcal{B}_X\} dg = O(X^{\dim V - \delta}).$$

Proof. It suffices to do so in each cusp separately, so fix a cusp corresponding to α_j. The region of integration in this case is a subset of

$$\alpha_j \omega_j \{t \in A'_c \mid |\alpha(t)| \geq c' \text{for some } \alpha \in S_G\} K_1,$$

for some choice of c, c'. The computations in the proof of Proposition 5.3 directly show that the integral in this case is $O(X^{\dim V - \delta})$ for some $\delta > 0$, as wanted. □

5.2 Slicing

The results in Section 5.1 show that

$$N(\Gamma, V_i, X) = \frac{1}{r_i \operatorname{vol}(G_0)} \sum_{j=1}^{m} \int_{g \in \alpha_j \omega_j A'_c K_1} \#\{v \in (\alpha_j \cdot W_0) \cap g\mathcal{B}_X\} dg + O(X^{\dim V - \delta}) \tag{7}$$

for some constant $\delta > 0$. To estimate the number of lattice points in the given region, we would want to use Proposition 5.2. However, we can't use it directly, because some of the projections onto coordinate hyperplanes of W_0 are of the order of the main term! To circumvent this, we will slice the region W_0 according to the values of the height-one coefficients.

We will compute the integral in (7) separately for each cusp $\alpha_j \omega_j A'_c K_1$. To simplify notation, we will work with the cusp with $\alpha_j = 1$, since the computations for the rest are analogous.

Let $v \in W_0(F_\infty)$, and let $v_1, \dots, v_k$ denote its height-one coefficients, where $k = \#S_G$. For any $b \in (F_\infty)^k$ and any subset $\mathcal{S} \subset W_0(F_\infty)$, we write

$$\mathcal{S}_b = \{v \in \mathcal{S} \mid (v_1, \dots, v_k) = b\}.$$

For any $v \in V_0 \cap W_0$, we know that the values of its height-one coefficients fall into some lattice $\mathcal{L} \subset F^k$ which as an additive subgroup is commensurable to $\mathcal{O}_F^k$. We can then write

$$\#(g\mathcal{B}_X \cap W_0 \cap V_0) = \sum_{b \in \mathcal{L}} \#(g\mathcal{B}_X \cap W_0 \cap V_0)_b$$

In fact, we can avoid summing over some of the b:

Lemma 5.6. *Let $v \in W_0(F_\infty)$. If $v_i = 0$ for some height-one coefficient i, then $\Delta(v) = 0$.*

Proof. It suffices to consider each completion F_w separately, for all infinite places w. Let $\{\alpha_1, \dots, \alpha_k\}$ be the height-one weights, and assume that the coefficient of α_i of v is zero in F_v. Let $\lambda_i \colon \mathbb{G}_m \to G_{\mathbb{C}}$ be the one-parameter subgroup such that $(\alpha_j \circ \lambda_i)(t) = t^{\delta_{ij}}$. Then, v has no positive weights with respect to λ_i, and so by Proposition 2.2 we get the result. □

We can now use Davenport's lemma, using the natural bijection between F_∞ and $\mathbb{R}^{[F:\mathbb{Q}]}$, to estimate

$$\#(g\mathcal{B}_X \cap W_0 \cap V_0)_b = \mathrm{vol}((g\mathcal{B}_X)|_b)(1 + O(X^{-1/[F:\mathbb{Q}]})).$$

The weight of any non-height-one coefficient in W_0 is $\gg 1$, so the range of values of any real coefficient varying in $\mathcal{B}_X$ is $\gg X^{1/[F:\mathbb{Q}]}$, so we get an error term of the order of $O(X^{-1/[F:\mathbb{Q}]})$.

Now, note that $KG_0 = G_0$ and that unipotent transformations preserve both the value of the height-one coefficients and the volume, so we get that $\mathrm{vol}((ntk\mathcal{B}_X)|_b) = \mathrm{vol}((t\mathcal{B}_X)|_b)$, where $t \in A'_c$. It will be convenient for us to integrate not over A'_c, but rather over $T \cap \mathcal{S}_1$, which is a set of the form $A'_c \times K_T$, for some subset $K_T \subset K_1$. We note that by construction of K_1, the set $A'_c \times K_T$ corresponds to a fundamental domain for the action of $\Gamma \cap T$ on $T(F_\infty)$ (cf. Section 4.3.1). There is a natural measure on $A'_c \times K_T$, inherited by restriction of the measures dt and dk: we will denote it dt by abuse of notation. Then, given that ω_j and K_1 have finite measure, we get that

$$N(\Gamma, V_i, X) = C \sum_{\substack{b \in \mathcal{L} \\ b_i \neq 0 \,\forall i}} \int_{t \in A'_c \times K_T} \mathrm{vol}((t\mathcal{B}_X)|_b) |\delta(t)|^{-1} dt, \tag{8}$$

for some constant C. For each height-one coefficient v_i, its weight under the action of T is $\alpha_i(t)$. We will let $\beta_i := b_i/(X^{1/[F:\mathbb{Q}]}\alpha_i(t))$, and $\beta = (\beta_i)_i$. Denote by Φ_V the different weights of the action of T on V, and by Φ_V^- the negative weights. Then, we have that

$$\mathrm{vol}((t\mathcal{B}_X)|_b) = \mathrm{vol}(tX^{1/[F:\mathbb{Q}]} \cdot (\mathcal{B}_1)|_\beta) = X^{\dim W_b} \prod_{\alpha \in \Phi_V^-} |\alpha(t)| \,\mathrm{vol}(\mathcal{B}|_\beta).$$

We will make the change of variables $t \mapsto \beta = (\beta_1, \dots, \beta_k)$, under which $dt = d\beta$, where $d\beta = \prod_{i=1}^k \frac{d\beta_i}{|\beta_i|}$. In Section 5.4, we will explicitly compute the volume of the cuspidal region for each of the possible cases. We will obtain a polynomial $Z(\beta) = \prod_i \beta_i^{e_i}$ with integer exponents $e_i \geq 2$, and we will see that

$$X^{\#\Phi_V^-} \prod_{\alpha \in \Phi_V^-} |\alpha(t)||\delta^{-1}(t)| = X^{\dim V} \frac{|Z(\beta)|}{|Z(b)|}. \tag{9}$$

The group $\Gamma_T := \Gamma \cap T$ acts naturally on $\mathcal{L}$ by $t \cdot (b_1, \ldots, b_k) = (\alpha_1(t)b_1, \ldots, \alpha_k(t)b_k)$ for $t \in \Gamma_T$ and $b \in \mathcal{L}$ (we know that $t \cdot b \in \mathcal{L}$ because V_0 is invariant under Γ). Denote $A' = \cup_{c>0} A'_c$ (i.e. the region defined by A'_c without the condition that $|\alpha(t)| \leq c$). The change of variables from t to β sends the domain of integration $A' \times K_T$ to some region $Y_b \subset F_\infty^{|M_\infty|}$, and say that the region $A'_c \times K_T$ gets sent to $Y_b \setminus Y_c$. It follows that

$$\int_{t \in A'_c \times K_T} \operatorname{vol}((t\mathcal{B}_X)|_b)|\delta^{-1}(t)|dt = \frac{X^{\dim V}}{|Z(b)|} \int_{\beta \in Y_b \setminus Y_c} |Z(\beta)| \operatorname{vol}(\mathcal{B}|_\beta) d\beta.$$

The set Y_c corresponds to elements β with $|\beta| \gg X^{-1}$ – in particular, the integral over Y_c is bounded by $O(X^{-1})$, so it can be added to the error term. Now, for any $b_0 \in \mathcal{L}$, we have that at most $O_\varepsilon(X^\varepsilon)$ choices of $b \in \Gamma_t b_0$ give a non-zero volume of $\mathcal{B}|_\beta$. Therefore, we can write

$$\sum_{b \in \Gamma_T b_0} \int_{t \in A'_c \times K_T} \operatorname{vol}((t\mathcal{B}_X)|_b)|\delta^{-1}(t)|dt = \sum_{b \in \Gamma_T b_0} \frac{X^{\dim V}}{|Z(b)|} \int_{\beta \in Y_b} |Z(\beta)| \operatorname{vol}(\mathcal{B}|_\beta) d\beta + O_\varepsilon(X^{-1+\varepsilon}).$$

Now, $\cup_{b \in \Gamma_T b_0} Y_b = (F_\infty^{|M_\infty|})^k$, so we get:

$$\begin{aligned} \sum_{b \in \Gamma_T b_0} \int_{t \in A'_c \times K_T} \operatorname{vol}((t\mathcal{B}_X)|_b)|\delta^{-1}(t)|dt &= \frac{X^{\dim V}}{|Z(b_0)|} \int_{\beta \in (F_\infty^{|M_\infty|})^k} |Z(\beta)| \operatorname{vol}(\mathcal{B}|_\beta) d\beta + O_\varepsilon(X^{-1+\varepsilon}) \\ &= \frac{X^{\dim V}}{|Z(b_0)|} \int_{v \in W_0} |Z^\times(v)| dv + O_\varepsilon(X^{-1+\varepsilon}). \end{aligned}$$

Here, $Z^\times(\beta) = \prod_{i=1}^k \beta_i^{e_i - 1}$, where the e_i are the exponents appearing in $Z(\beta)$. Adding over all $b_0 \in \mathcal{L}/\Gamma_T$ and combining with (8), we conclude the proof of Theorem 5.1.

5.3 Congruence conditions

In the sequel, it will be convenient for us to not only have an estimate for the number of reducible orbits, but we will also need some knowledge about what happens when we impose finitely many congruence conditions in our orbits. It will suffice to do our analysis in the cusp; to that effect, consider the counting function

$$N^{cusp}(\Gamma, V_0, X) = \int_{g \in \mathcal{F}} \#\{v \in V_0 \cap W_0 \cap (g\mathcal{B}_X)\} dg.$$

In the previous sections, we obtained that

$$N^{cusp}(\Gamma, V_0, X) = CX^{\dim V} + O(X^{\dim V - \frac{1}{[F:\mathbb{Q}]}}),$$

where C and the implied constant depend only on the choice of Γ and V_0. We will now obtain the following:

Theorem 5.7. *Let L be a translate of IV_0, for some ideal I of $\mathcal{O}_F$. If*

$$N^{cusp}(\Gamma, V_0, X) = CX^{\dim V} + O(X^{\dim V - \frac{1}{[F:\mathbb{Q}]}}),$$

then

$$N^{cusp}(\Gamma, L, X) = C(NI)^{-\dim V} X^{\dim V} + O((NI)^{\frac{1}{[F:\mathbb{Q}]} - \dim V} X^{\dim V - \frac{1}{[F:\mathbb{Q}]}}).$$

The implied constant is independent of the choice of L.

Proof. The argument goes through in the same way as Section 5.2, and the only difference is in the application of Davenport's lemma, where the additional terms appear by taking care of the change of lattices. □

5.4 Case-by-case analysis

In this section, we complete the proof of Theorem 5.1 by performing a case-by-case analysis. For the D_n and E_n cases, we will explicitly compute the dimension and volume of $W_\flat$ (which was defined to be the set of coefficients of W_0 of non-positive height), and the modular function $\delta(t) = \prod_{\beta\in\Phi_G^-}\beta(t) = \det \mathrm{Ad}(t)|_{\mathrm{Lie}\,N(F_\infty)}$.

5.4.1 D_{2n+1}

The exposition in the D_n cases is inspired by [Lag22a, Appendix A] and [Sha18, §7.2.1]. We start by describing explicitly the representation (G,V) of D_{2n+1} in the form given by Table 1.

Let $n\geq 2$ be an integer. Let U_1 be a $\mathbb{Q}$-vector space with basis $\{e_1,\ldots,e_n,u_1,e_n^*,\ldots,e_1^*\}$, endowed with the symmetric bilinear form b_1 satisfying $b_1(e_i,e_j) = b_1(e_i,u_1) = b_1(e_i^*,e_j^*) = b_1(e_i^*,u_1) = 0$, $b_1(e_i,e_j^*) = \delta_{ij}$ and $b_1(u_1,u_1) = 1$ for all $1\leq i,j\leq n$. In this case, given a linear map $A\colon U\to U$ we can define its *adjoint* as the unique map $A^*\colon U\to U$ satisfying $b_1(Av,w) = b_1(v,A^*w)$ for all $v,w\in U$. In terms of matrices, A^* corresponds to taking the reflection of A along its antidiagonal when working with the fixed basis. We can define $\mathrm{SO}(U_1,b_1) := \{g\in \mathrm{SL}(U_1)\mid gg^*=\mathrm{id}\}$, with a Lie algebra that can be identified with $\{A\in\mathrm{End}(U)\mid A=-A^*\}$.

Let U_2 be a $\mathbb{Q}$-vector space with basis $\{f_1,\ldots,f_n,u_2,f_n^*,\ldots,f_1^*\}$, with a similarly defined bilinear form b_2. Let $(U,b) = (U_1,b_1)\oplus(U_2,b_2)$. Let $H=\mathrm{SO}(U,b)$, and consider $\mathfrak{h} := \mathrm{Lie}\,H$. With respect to the basis

$$\{e_1,\ldots,e_n,u_1,e_n^*,\ldots,e_1^*,f_1,\ldots,f_n,u_2,f_n^*,\ldots,f_1^*\},$$

the adjoint of a block matrix according to the bilinear form b is given by

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}^* = \begin{pmatrix} A^* & C^*\\ B^* & D^*\end{pmatrix},$$

where A^*,B^*,C^*,D^* denote reflection by the antidiagonal. An element of $\mathfrak{h}$ is given by

$$\left\{\begin{pmatrix} B & A\\ -A^* & C\end{pmatrix}\,\middle|\, B=-B^*,\, C=-C^*\right\}.$$

The stable involution θ is given by conjugation by $\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$, where the first $2n+1$ entries are 1 and the last $2n+1$ entries are given by -1. Under this description, we see that

$$V = \left\{\begin{pmatrix} 0 & A\\ -A^* & 0\end{pmatrix}\,\middle|\, A\in\mathrm{Mat}_{(2n+1)\times(2n+1)}\right\}.$$

Moreover, $G=(H^\theta)^\circ$ is isomorphic to $\mathrm{SO}(U_1)\times\mathrm{SO}(U_2)$. We will use the map

$$\begin{pmatrix} 0 & A\\ -A^* & 0\end{pmatrix}\mapsto A$$

to establish a bijection between V and $\mathrm{Hom}(U_2,U_1)$, where $(g,h)\in\mathrm{SO}(U_1)\times\mathrm{SO}(U_2)$ acts on $A\in V$ as $(g,h)\cdot A = gAh^{-1}$.

Let T be the maximal torus $\mathrm{diag}(t_1,\ldots,t_n,1,t_n^{-1},\ldots,t_1^{-1},s_1,\ldots,s_n,1,s_n^{-1},\ldots,s_1^{-1})$ of G. A basis of simple roots for G is

$$S_G = \{t_1-t_2,\ldots,t_{n-1}-t_n\}\cup\{s_1-s_2,\ldots,s_{n-1}-s_n\}.$$

A positive root basis for V can be taken to be

$$S_V = \{t_1-s_1, s_1-t_2,\ldots,t_n-s_n,s_n\}.$$

For convenience, we now switch to multiplicative notation for the roots. We make the change of variables $\alpha_i = t_i/t_{i+1}$ for $i = 1, \dots, n-1$ and $\alpha_n = t_n$; similarly $\gamma_i = s_i/s_{i+1}$ for $i = 1, \dots, n-1$ and $\gamma_n = s_n$. The estimate for the volume becomes

$$\prod_{\lambda \in \Phi_V^-} X|\lambda(t)| = X^{2n^2+2n+1} \prod_{i=1}^{n} |\alpha_i|^{-2in+i^2-2i} |\gamma_i|^{-2in+i^2}.$$

The modular function in our case is

$$|\delta^{-1}(t)| = \prod_{i=1}^{n} |\alpha_i|^{2in-i^2} |\gamma_i|^{2in-i^2}.$$

Changing variables to $\beta_i = b_i/(X\lambda_i(v_i))$, where λ_i are the height-one weights, we obtain

$$\prod_{\lambda \in \Phi_V^-} X|\lambda(t)|\delta^{-1}(t) = X^{4n^2+4n+1} \frac{|Z(\beta)|}{|Z(b)|},$$

where $Z(\beta) := \prod_{i=1}^{n} (\beta_{2i-1}\beta_{2i})^{2i}$.

5.4.2 D_{2n}

The analysis in this case is very similar to the D_{2n+1} case. Now, we consider the $\mathbb{Q}$-vector space U_1 with basis $\{e_1, \dots, e_n, e_n^*, \dots, e_1^*\}$, endowed with a symmetric bilinear form $b_1(e_i, e_j) = b_1(e_i^*, e_j^*) = 0$, $b_1(e_i, e_j^*) = \delta_{ij}$. We also consider a $\mathbb{Q}$-vector space U_2 with basis $\{f_1, \dots, f_n, f_n^*, \dots, f_1^*\}$, with an analogous symmetric bilinear form b_2.

Let $(U, b) = (U_1, b_1) \oplus (U_2, b_2)$, let $H' = \mathrm{SO}(U, b)$ and define H to be the quotient of H' by its centre of order 2. Under the basis

$$\{e_1, \dots, e_n, e_n^*, \dots, e_1^*, f_1, \dots, f_n, f_n^*, \dots, f_1^*\},$$

the stable involution is given by conjugation with $\mathrm{diag}(1, \dots, 1, -1, \dots, -1)$. Similarly to the D_{2n+1} case, we have

$$V = \left\{ \begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix} \middle| A \in \mathrm{Mat}_{2n \times 2n} \right\},$$

where A^* denotes reflection by the antidiagonal. In this case, the group $G = (H^\theta)^\circ$ is isomorphic to $\mathrm{SO}(U_1) \times \mathrm{SO}(U_2)/\Delta(\mu_2)$, where $\Delta(\mu_2)$ denotes the diagonal inclusion of μ_2 into the centre $\mu_2 \times \mu_2$ of $\mathrm{SO}(U_1) \times \mathrm{SO}(U_2)$. As before, we can identify V with the space of $2n \times 2n$ matrices using the map

$$\begin{pmatrix} 0 & A \\ -A^* & 0 \end{pmatrix} \mapsto A,$$

where $(g, h) \in G$ acts by $(g, h) \cdot A = gAh^{-1}$.

We consider the maximal torus T of H given by $\mathrm{diag}(t_1, \dots, t_n, t_n^{-1}, \dots, t_1^{-1}, s_1, \dots, s_n, s_n^{-1}, \dots, s_1^{-1})$. A basis of simple roots for H and G are given by

$$S_H = \{t_1 - s_1, s_1 - t_2, \dots, s_{n-1} - t_n, t_n - s_n, s_n + t_n\},$$
$$S_G = \{t_1 - t_2, \dots, t_{n-1} - t_n, t_{n-1} + t_n\} \cup \{s_1 - s_2, \dots, s_{n-1} - s_n, s_{n-1} + s_n\}.$$

Let $\alpha_i = t_i/t_{i+1}$ and $\gamma_i = s_i/s_{i+1}$ for $i = 1, \dots, n$, and let $\alpha_n = t_{n-1}t_n$ and $\gamma_n = s_{n-1}s_n$. Under this change of variables, the estimate for the volume is:

$$\prod_{\lambda \in \Phi_V^-} X|\lambda(t)| = X^{2n^2} \left(\prod_{i=1}^{n-2} |\alpha_i|^{-2in+i^2-i} |\alpha_{n-1}|^{(-n^2-n+4)/2} |\alpha_n|^{(-n^2-n)/2} \prod_{i=1}^{n-2} |\gamma_i|^{-2in+i^2+i} |\gamma_{n-1}\gamma_n|^{(-n^2+n)/2} \right).$$

The modular function is

$$|\delta^{-1}(t)| = \prod_{i=1}^{n-2} |\alpha_i\gamma_i|^{i^2-2in+i} |\alpha_{n-1}\gamma_{n-1}\alpha_n\gamma_n|^{-(n-1)n/2}.$$

As before, we can compute:

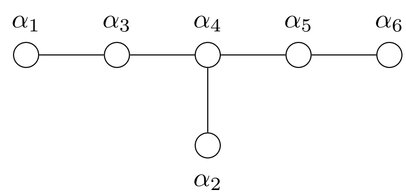
$$\prod_{\lambda\in\Phi_V^-} X|\lambda(t)||\delta^{-1}(t)| = X^{4n^2}\frac{|Z(\beta)|}{|Z(b)|},$$

where $Z(\beta) = \prod_{i=1}^{n-1}(\beta_{2i-1}\beta_{2i})^{2i} \cdot (\beta_{2n-1}\beta_{2n})^n$.

5.4.3 E_6

For the E_6 case, we use the conventions and computations in [Tho15, §2.3, §5].

Let $S_H = \{\alpha_1, \dots, \alpha_6\}$, where the Dynkin diagram of H is:



The pinned automorphism ϑ consists of a reflection around the vertical axis. We can define a root basis $S_G = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ of G as $\gamma_1 = \alpha_3 + \alpha_4$, $\gamma_2 = \alpha_1$, $\gamma_3 = \alpha_3$ and $\gamma_4 = \alpha_2 + \alpha_4$. Under this basis, we have

$$\prod_{\lambda\in\Phi_V^-} X|\lambda(t)| = X^{22}|\gamma_1|^{-12}|\gamma_2|^{-18}|\gamma_3|^{-22}|\gamma_4|^{-12}$$

The modular function is

$$|\delta^{-1}(t)| = |\gamma_1|^8|\gamma_2|^{14}|\gamma_3|^{18}|\gamma_4|^{10}.$$

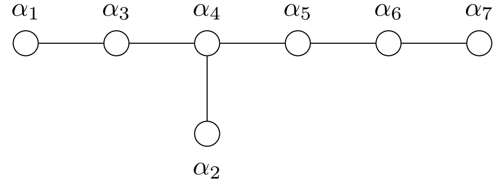
The weights of the height-one coefficients are $\{\gamma_2, -\gamma_1+\gamma_3+\gamma_4, \gamma_3, \gamma_1-\gamma_3\}$. In light of this, we obtain

$$\prod_{\lambda\in\Phi_V^-} X|\lambda(t)||\delta^{-1}(t)| = X^{42}\frac{|Z(\beta)|}{|Z(b)|}.$$

where $Z(\beta) = \beta_1^4\beta_2^2\beta_3^8\beta_4^6$.

5.4.4 E_7

For the E_7 and E_8 cases, we follow the conventions in [RT18]. Let $S_H = \{\alpha_1, \dots, \alpha_7\}$, where the Dynkin diagram of H is:



The root basis $S_G = \{\gamma_1, \ldots, \gamma_7\}$ can be described as

$$\begin{aligned}
\gamma_1 &= \alpha_3 + \alpha_4 \\
\gamma_2 &= \alpha_5 + \alpha_6 \\
\gamma_3 &= \alpha_2 + \alpha_4 \\
\gamma_4 &= \alpha_1 + \alpha_3 \\
\gamma_5 &= \alpha_4 + \alpha_5 \\
\gamma_6 &= \alpha_6 + \alpha_7 \\
\gamma_7 &= \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5
\end{aligned}$$

The volume of $W_\flat$ can be computed to be

$$\prod_{\lambda \in \Phi_V^-} X|\lambda(t)| = X^{35}|\gamma_1|^{-15/2}|\gamma_2|^{-13}|\gamma_3|^{-33/2}|\gamma_4|^{-18}|\gamma_5|^{-35/2}|\gamma_6|^{-15}|\gamma_7|^{-21/2}.$$

The modular function for G can be computed to be

$$\delta^{-1}(t) = |\gamma_1|^7|\gamma_2|^{12}|\gamma_3|^{15}|\gamma_4|^{16}|\gamma_5|^{15}|\gamma_6|^{12}|\gamma_7|^7.$$

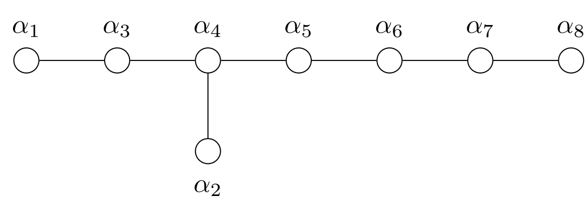
We can compute the weights β_i corresponding to the height-one coefficients, with the end result being

$$\prod_{\lambda \in \Phi_V^-} X|\lambda(t)||\delta^{-1}(t)| = X^{70}\frac{|Z(\beta)|}{|Z(b)|},$$

for $Z(\beta) = \beta_1^2\beta_2^5\beta_3^6\beta_4^8\beta_5^7\beta_6^4\beta_7^3$.

5.4.5 E_8

Let $S_H = \{\alpha_1, \ldots, \alpha_8\}$, where the Dynkin diagram of H is:



The root basis $S_G = \{\gamma_1, \ldots, \gamma_8\}$ can be described as

$$\begin{aligned}
\gamma_1 &= \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 \\
\gamma_2 &= \alpha_6 + \alpha_7 \\
\gamma_3 &= \alpha_4 + \alpha_5 \\
\gamma_4 &= \alpha_1 + \alpha_3 \\
\gamma_5 &= \alpha_2 + \alpha_4 \\
\gamma_6 &= \alpha_5 + \alpha_6 \\
\gamma_7 &= \alpha_7 + \alpha_8 \\
\gamma_8 &= \alpha_3 + \alpha_4
\end{aligned}$$

The volume of $W_\flat$ can be computed to be

$$\prod_{\lambda\in\Phi_V^-} X|\lambda(t)| = X^{64}|\gamma_1|^{-18}|\gamma_2|^{-30}|\gamma_3|^{-40}|\gamma_4|^{-48}|\gamma_5|^{-54}|\gamma_6|^{-58}|\gamma_7|^{-30}|\gamma_8|^{-30}.$$

The modular function for G can be computed to be

$$|\delta^{-1}(t)| = |\gamma_1|^{14}|\gamma_2|^{26}|\gamma_3|^{36}|\gamma_4|^{44}|\gamma_5|^{50}|\gamma_6|^{54}|\gamma_7|^{28}|\gamma_8|^{28}.$$

We get

$$\prod_{\lambda\in\Phi_V^-} X|\lambda(t)||\delta^{-1}(t)| = X^{128}\frac{|Z(\beta)|}{|Z(b)|},$$

with $Z(\beta) = \beta_1^4\beta_2^8\beta_3^{10}\beta_4^{14}\beta_5^{12}\beta_6^8\beta_7^6\beta_8^2$.

6 Proof of the main results

We are now in a position to prove the main results.

6.1 Elements with big stabiliser

We first proof a necessary result about elements with a big stabiliser. As in Section 5, let V_0 be a lattice inside $V(F_\infty)$ which is commensurable with $V(\mathcal{O}_K)$, and let Γ be an arithmetic subgroup of $G(F)$ which preserves V_0. We denote by V_0^{bs} the set of elements $v \in V_0$ which satisfy $\#\operatorname{Stab}_{G(F)} v > 1$.

Proposition 6.1. *There exists a constant $\delta_{bs} > 0$ with*

$$N(\Gamma, V_0^{bs,red}, X) = O(X^{\dim V - \delta_{bs}}).$$

Proof. We can see that the density of elements in $V(\mathcal{O}_\mathfrak{p})$ having big stabiliser tends to some constant in $(0, 1)$ by the same argument as in [Lag22a, Proof of Lemma 8.20]. Then, we can use the Selberg sieve as in Proposition 5.4, now using the estimate in Theorem 5.1. □

Remark 6.2. We remark that we could not have proven Proposition 6.1 at the same time as Proposition 5.4, as we need the estimate in Theorem 5.1 to apply the Selberg sieve in this case.

6.2 Tail estimates

To prove Theorem 1.2, we need to obtain tail estimates for the sets $\mathcal{W}_I^{(1)}$ and $\mathcal{W}_I^{(2)}$. The required estimate for $\mathcal{W}_I^{(1)}$ can be obtained using [BSW15, Theorem 18] (which is essentially by following the argument in [Bha14b, Theorem 3.3]). For $\mathcal{W}_I^{(2)}$, recall that we only deal with ideals I which are coprime to a certain element N_{bad}, as explained in Section 3. We can prove the following:

Theorem 6.3. *There exists a constant $\delta > 0$ such that*

$$\sum_{\substack{I \text{ squarefree} \\ NI > M \\ (I, N_{bad})=1}} \#\{b \in \mathbb{G}_m(F)\backslash\mathcal{W}_I^{(2)} \mid \operatorname{ht}(b) < X\} = O\left(\frac{X^{\dim V}}{M}\right) + O(X^{\dim V - \delta}).$$

Proof. In Section 3, for every $\beta_i \in \mathrm{cl}(G)$ we constructed

$$W_{i,M} := \{v \in V_{\beta_i} \mid v = g_I \kappa_b,\, I \text{ squarefree},\, I \text{ coprime to } N_{bad},\, NI > M,\, g_I \in G_I,\, b \in B(\mathcal{O}_K)\}.$$

By the results in Section 3, it is sufficient to obtain an appropriate bound on the number of G_{β_i}-orbits of $W_{i,M}$ with invariants in Σ. By following the same averaging argument as in Section 5, we get that

$$N(G_{\beta_i}, W_{i,M}, X) = \int_{g\in\mathcal{F}} \#\{v \in W_{i,M} \cap g\mathcal{B}_X\} dg.$$

We can carry out the same argument, now assuming that we can restrict to those elements in $W_{i,M}$ with trivial stabiliser by Proposition 6.1. If $v \in W_{i,M} \cap W_0$ has trivial stabiliser, then $Q(v) \gg M$ by Proposition 3.4, or alternatively $Z(v) \gg M^2$. We have that:

$$N(G_{\beta_i}, W_{i,M}, X) \ll \sum_{\substack{b_0\in\mathcal{L}/\Gamma_T\\ Z(b_0)\gg M^2}} \frac{1}{|Z(b_0)|} + O(X^{\dim V-\delta}),$$

and the written sum is $O(1/M)$, which concludes the proof. □

Thus, we have concluded the proof of Theorem 1.2. We can combine both estimates for the strongly divisible case and the weakly divisible case. For I a squarefree ideal, denote by $\mathcal{W}_I$ the set of elements $b \in B(\mathcal{O}_F)$ such that I^2 divides $\Delta(b)$. Then, in the style of [BSW22a, Theorem 4.4], we obtain:

Theorem 6.4. *There is a constant $\delta > 0$ such that*

$$\sum_{\substack{I \text{ squarefree}\\ NI>M\\ (I,N_{bad})=1}} \#\{b \in \mathbb{G}_m(F)\backslash\mathcal{W}_I \mid \mathrm{ht}(b) < X\} = O_\varepsilon\left(\frac{X^{\dim V+\varepsilon}}{\sqrt{M}}\right) + O(X^{\dim V-\delta}).$$

6.3 A squarefree sieve

Theorem 1.1 follows from Theorem 1.2 by using a squarefree sieve. In fact, we will prove a slightly more general result about families in $\Sigma \subset B(\mathcal{O}_F)$ defined by infinitely many congruence conditions.

Let κ be a positive integer. We will say that $\mathcal{S} \subset \Sigma$ is *κ-acceptable* if $\mathcal{S} = \cap_{\mathfrak{p}\text{ finite}}\mathcal{S}_\mathfrak{p}$, where $\mathcal{S}_\mathfrak{p} \subset \Sigma_\mathfrak{p} \subset B(\mathcal{O}_\mathfrak{p})$ satisfy the following:

- $\mathcal{S}_\mathfrak{p}$ is defined by congruence conditions modulo $\mathfrak{p}^\kappa$.
- For all sufficiently large primes $\mathfrak{p}$, the set $\mathcal{S}_\mathfrak{p}$ contains all elements with $\mathfrak{p}^2$ not dividing $\Delta(b)$.

For any subset $A \subset \Sigma$, we denote by $N(A, X)$ the number of elements of A having height less than X. For any prime $\mathfrak{p}$ and any subset $A_\mathfrak{p} \subset \Sigma_\mathfrak{p}$, we denote by $\rho(A_\mathfrak{p})$ the density of $A_\mathfrak{p}$ inside $\Sigma_\mathfrak{p}$.

Theorem 6.5. *Let $\mathcal{S} = \cap_\mathfrak{p}\mathcal{S}_\mathfrak{p}$ be a κ-acceptable subset of Σ. Then, there exists a constant $\delta > 0$ such that*

$$N(\mathcal{S}, X) = \prod_\mathfrak{p} \rho(\mathcal{S}_\mathfrak{p}) N(\Sigma, X) + O(X^{\dim V-\delta}).$$

Proof. Recall that $B = \operatorname{Spec} \mathcal{O}_F[p_{d_1}, \ldots, p_{d_k}]$. For an element $b \in \Sigma$, we have that $|p_{d_i}(b)| \ll X^{d_i}$, where by Table 1 we see that $d_i \geq 2$ for all i. For I a squarefree ideal coprime to N_{bad}, we define a family $\mathcal{S}^I = \cap_\mathfrak{p}\mathcal{S}^I_\mathfrak{p}$ as follows:

- If $\mathfrak{p} \mid N_{bad}$, then $\mathcal{S}_{\mathfrak{p}}^I = \mathcal{S}_{\mathfrak{p}}$.
- If $\mathfrak{p} \mid I$, then $\mathcal{S}_{\mathfrak{p}}^I = \Sigma_{\mathfrak{p}} \setminus \mathcal{S}_{\mathfrak{p}}$.
- Otherwise, set $\mathcal{S}_{\mathfrak{p}}^I = \Sigma_{\mathfrak{p}}$.

By the inclusion-exclusion principle, we have

$$N(\mathcal{S}, X) = \sum_{\substack{I \text{ squarefree} \\ (I, N_{bad})=1}} \mu(I) N(\mathcal{S}^I, X),$$

where μ is the Möbius function for the ideals of $\mathcal{O}_F$. By the Chinese Remainder Theorem, we can estimate

$$N(\mathcal{S}^I, X) = \prod_{\mathfrak{p} \mid N_{bad}} \rho(\mathcal{S}_{\mathfrak{p}}) \prod_{\mathfrak{p} \mid I} (1 - \rho(\mathcal{S}_{\mathfrak{p}})) N(\Sigma, X) + O((NI)^{\kappa} X^{\dim V - 2})$$

From Theorem 6.4, we also know that

$$\sum_{\substack{I \text{ squarefree} \\ (I, N_{bad})=1 \\ NI > M}} \mu(I) N(\mathcal{S}^I, X) = O_{\varepsilon}\left(\frac{X^{\dim V + \varepsilon}}{M}\right) + O(X^{\dim V - \delta}).$$

Thus, we get that

$$\begin{aligned} N(\mathcal{S}^I, X) &= \prod_{\mathfrak{p} \mid N_{bad}} \rho(\mathcal{S}_{\mathfrak{p}}) N(\Sigma, X) \sum_{\substack{I \text{ squarefree} \\ (I, N_{bad})=1 \\ NI \le M}} \mu(I) \prod_{\mathfrak{p} \mid I} (1 - \rho(\mathcal{S}_{\mathfrak{p}})) + O_{\varepsilon}\left(\frac{X^{\dim V + \varepsilon}}{\sqrt{M}} + X^{\dim V - \delta} + M^{\kappa+1} X^{\dim V - 2}\right) \\ &= \prod_{\mathfrak{p}} \rho(\mathcal{S}_{\mathfrak{p}}) N(\Sigma, X) + O_{\varepsilon}\left(\frac{X^{\dim V}}{M} + \frac{X^{\dim V + \varepsilon}}{\sqrt{M}} + X^{\dim V - \delta} + M^{\kappa+1} X^{\dim V - 2}\right). \end{aligned}$$

Optimizing, we choose $M = X^{4/(2\kappa+3)}$ and we conclude the proof. □

A Counting irreducible orbits

In the proof of Proposition 5.4, we need a power-saving asymptotic for the number of orbits in the main body. As in Section 5, we let Γ be an arithmetic subgroup of $G(F)$, and we let V_0 be a lattice of $V(F)$ which is commensurable with $V(\mathcal{O}_F)$ and Γ-stable. We denote by $N^*(\Gamma, V_0, X)$ the number of Γ-orbits in V_0 with invariants in $\Lambda\Sigma$ having height at most X (recall that Λ is the image of the natural embedding of $\mathbb{R}_{>0}$ inside F_{∞}). Then, we prove the following result:

Theorem A.1. *There exist positive constants C, δ such that*

$$N^*(\Gamma, V_0, X) = C X^{\dim V} + O(X^{\dim V - \delta}).$$

The constant C depends only on Γ and V_0, while $\delta > 0$ can be chosen independently of Γ and V_0.

Proof. Following the notations in Section 5.1, we get that

$$N^*(\Gamma, V_0, X) = \frac{1}{r_i \operatorname{vol}(G_0)} \sum_{j=1}^{m} \int_{g \in \alpha_j \mathcal{S}_j} \#\{v \in V_0 \cap g\mathcal{B}_X \mid v_{0,j} \neq 0\} dg + O(X^{\dim V - \delta}).$$

It suffices to estimate the integral for each j separately: for simplicity, set $\alpha_j = 1$ and denote the highest weight as v_0. We then have that $|v_0| \geq C_0$ for some constant C_0. Let us denote $\mathcal{E} = \{v \in g\mathcal{B}_X \mid |v_0| \geq C_0\}$. There is a constant $J \geq 0$ such that any element in $\mathcal{B}_1$ satisfies $|v_0| \leq J$, so if $\mathcal{E}$ is non empty for some choice of $g = ntk$, we must have that $X|a_0(t)| \geq C_0/J$, where a_0 denotes the corresponding weight of v_0.

By Davenport's lemma (i.e. Proposition 5.2), we can approximate the number of lattice points in $\mathcal{E}$ by some constant times $\mathrm{vol}(\mathcal{E})$, with an error term corresponding to the volume of the lower-dimensional projections. The biggest volume of a lower-dimensional projection will correspond to setting one of the real coordinates of v_0 to be zero, and this volume can always be computed to be $X^{\dim V - \delta}$ for a suitable $\delta > 0$.

Then, given that $\mathcal{E} = g\mathcal{B}_X \setminus (g\mathcal{B}_X \setminus \mathcal{E})$ it remains to deal with

$$\int_{\substack{g=ntk\in\omega_j A'_c K_1\\ X|a_0(t)|\geq C_0/J}} (\mathrm{vol}(g\mathcal{B}_X) - \mathrm{vol}(g\mathcal{B}_X \setminus \mathcal{E}))dg$$

In the first summand, we note that $\mathrm{vol}(g\mathcal{B}_X)$ is independent of g, and that $\mathrm{vol}(\mathcal{B}_X) = X^{\dim V}\,\mathrm{vol}(\mathcal{B}_1)$, thus obtaining the main term. For the second summand, denote $\mathcal{E}' = g\mathcal{B}_X \setminus \mathcal{E}$. Any element in $v \in g\mathcal{B}_X \setminus \mathcal{E}$ must have $|v_0| \leq C_0$. Because of how the integration domain is set up, the values of v_0 fall in a compact region Ω of F_∞. For a given value of $v_0 \in \Omega$, let $\mathcal{E}'(v_0)$ denote the set of elements in $\mathcal{E}'$ with the given value of v_0. Then,

$$\mathrm{vol}(\mathcal{E}') = \int_{v_0\in\Omega} \mathrm{vol}(\mathcal{E}'(v_0)),$$

and each of the volumes of $\mathrm{vol}(\mathcal{E}'(v_0))$ can be computed to be $O(X^{\dim V - \delta})$, for some $\delta > 0$. □

The proof of Theorem A.1 immediately implies the following:

Corollary A.2. *Let L be a translate of IV_0 for some ideal $I \subset \mathcal{O}_F$. Then,*

$$N^*(\Gamma, L, X) = (NI)^{-\dim V} C X^{\dim V} + O((NI)^{-\dim V + \delta} X^{\dim V - \delta}).$$

Here, C and δ are as in Theorem A.1, and the implied constant is independent of the ideal I.

This result is what we need to apply the Selberg sieve in the proof of Proposition 5.4.

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