STOCHASTIC CALCULUS, LENT 2016, EXAMPLE SHEET 3

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Problem 1.

(a) Suppose that $(Z_t)_{t\geq 0}$ is a continuous local martingale which is strictly positive almost surely. Show that there is a unique continuous local martingale M such that $Z = \mathcal{E}(M)$, where

$$\mathcal{E}(M)_t = \exp(M_t - \tfrac{1}{2}[M]_t).$$

(b) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $\mathbb{Q}$ be another probability measure on $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ such that $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F}$. Show that if $Z_t = \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$ for all $0 \leq t \leq T$, then Z is a non-negative $\mathbb{P}$-martingale. Assuming that it is strictly positive almost surely and continuous, what can we say about the relationship between semimartingales with respect to $\mathbb{P}$ and $\mathbb{Q}$?

Problem 2. Let B be a standard Brownian motion and, for $a, b > 0$, let $\tau_{a,b} = \inf\{t \geq 0 : B_t + bt = a\}$. Use Girsanov's theorem to prove that the density of $\tau_{a,b}$ is given by

$$a(2\pi t^3)^{-1/2} \exp(-(a - bt)^2/2t).$$

Problem 3. Suppose that M is a continuous local martingale with $[M]_t \to \infty$ almost surely as $t \to \infty$. Show that $M_t/[M]_t \to 0$ as $t \to \infty$ and conclude that $\mathcal{E}(M)_t \to 0$ almost surely.

Problem 4. Suppose that X is a continuous local martingale with quadratic variation

$$[X]_t = \int_0^t A_s ds$$

for a non-negative, previsible process $(A_t)_{t\geq 0}$. Show that there exists a Brownian motion B (possibly defined on a larger probability space) such that

$$X_t = \int_0^t A_s^{1/2} dB_s.$$

Problem 5. Suppose that σ and b are Lipschitz. Explain why uniqueness in law holds for the SDE $dX_t = \sigma(X_t)dB_t + b(X_t)dt$.

Problem 6. A Bessel process of dimension δ is given by the solution to the SDE:

$$dX_t = \frac{\delta - 1}{2} \cdot \frac{1}{X_t} dt + dB_t, \quad X_0 > 0$$

where B is a standard Brownian motion, at least up until the first time t that $X_t = 0$.

(a) Show that $M_t = X_t^{2-\delta}$ is a continuous local martingale.

(b) For each a, let $\tau_a = \inf\{t \geq 0 : X_t = a\}$. For $a < X_0 < b$, compute $\mathbb{P}[\tau_a < \tau_b]$ using that M is a local martingale.

(c) Assume that $\delta < 2$. For $b > 1$, explain how one can condition on the event that $\tau_b < \tau_0$ using M.

(d) Using the previous part and the Girsanov theorem, describe the law of $X|_{[0,\tau_b]}$ conditioned on $\tau_b < \tau_0$.

(e) Explain why, informally, the statement "A standard Brownian motion conditioned to be positive is a 3-dimensional Bessel process" is true.

Problem 7. Suppose that $\mathbb{Q} \ll \mathbb{P}$. Show that if $X_n \to X$ in probability with respect to $\mathbb{P}$, then $X_n \to X$ in probability with respect to $\mathbb{Q}$.

Problem 8. Suppose that σ, b and σ_n, b_n for $n \in \mathbb{N}$ are Lipschitz with constant K uniformly in n. Suppose that $\sigma_n \to \sigma$ and $b_n \to b$ uniformly. Suppose that X and X^n are defined by

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x$$
$$dX_t^n = \sigma(X_t^n)dB_t + b_n(X_t^n)dt, \quad X_0^n = x.$$

Show for each $t > 0$ that

$$\mathbb{E}\left[\sup_{0 \le s \le t} |X_s^n - X_s|^2\right] \to 0 \quad \text{as} \quad n \to \infty.$$

Problem 9.

(a) Suppose that X is a weak solution of the SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$. Show that the process

$$f(X_t) - \int_0^t b(X_s)f'(X_s) + \frac{1}{2}\sigma^2(X_s)f''(X_s)ds$$

is a local martingale for all $f \in C^2$.

(b) Let X be a continuous, adapted process such that

$$f(X_t) - \int_0^t b(X_s)f'(X_s) + \frac{1}{2}\sigma^2(X_s)f''(X_s)ds$$

is a local martingale for each $f \in C^2$. Suppose σ is continuous and $\sigma(x) > 0$ for all x. Show that there exists a Brownian motion such that $dX_t = b(X_t)dt + \sigma(X_t)dW_t$. (Hint: use Problem 4.)

Problem 10. Let W be a standard Brownian motion.

(a) Let $B_t = W_t - tW_1$. Show that $(B_t)_{t \in [0,1]}$ is a continuous, mean-zero Gaussian process. What is the covariance $\mathbb{E}[B_s B_t]$?

(b) Is B adapted to the filtration generated by W?

(c) Let

$$dX_t = -\frac{X_t}{1-t}dt + dW_t, \quad X_0 = 0.$$

Verify that

$$X_t = (1-t)\int_0^t \frac{dW_s}{1-s} \quad \text{for} \quad 0 \le t < 1.$$

Show that $X_t \to 0$ as $t \uparrow 1$.

(d) Show that X is a continuous, mean-zero Gaussian process with the same covariance as B, i.e., X is a Brownian bridge.