Convergence of the SAW on random quadrangulations to $\mathrm{SLE}_{8/3}$ on $\sqrt{8/3}$-Liouville quantum gravity

Jason Miller

Cambridge

Ewain Gwynne (MIT)

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Outline

Part I: Planar maps

- Self-avoiding walks (SAW)
- SAWs on random planar maps
- Main scaling limit result

Part II: Liouville quantum gravity

- As a scaling limit / metric space
- Main gluing result

Part III: Proof ideas

Part I: Planar maps

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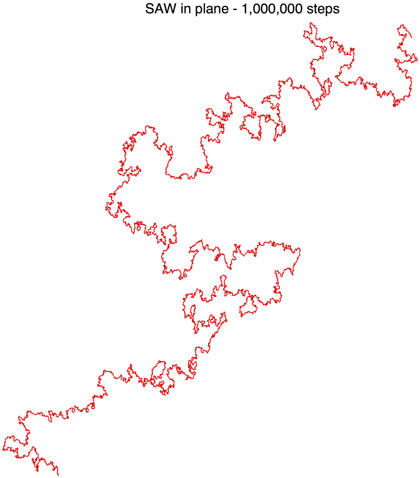

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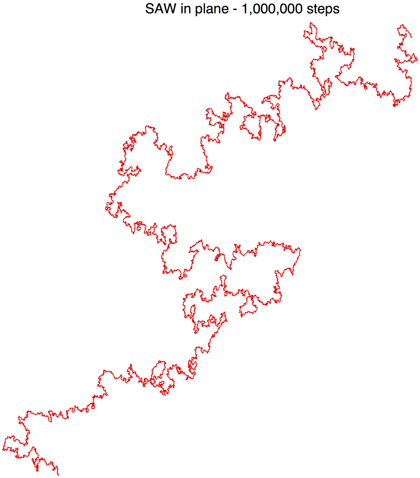

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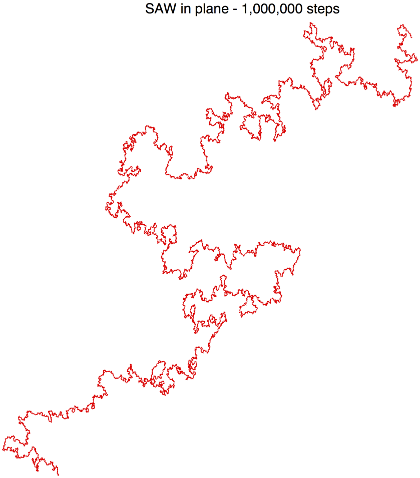

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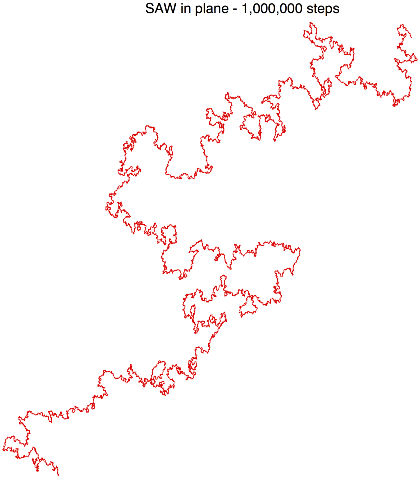

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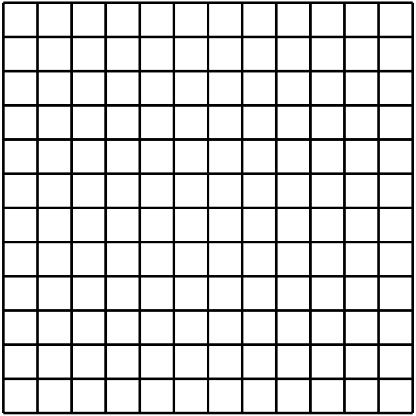

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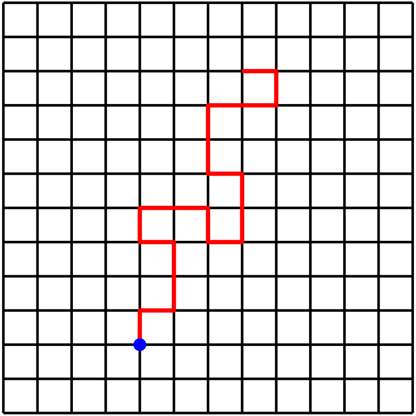

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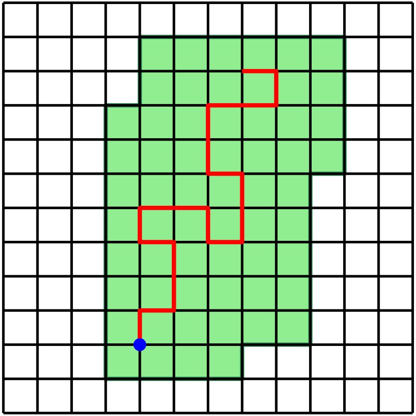

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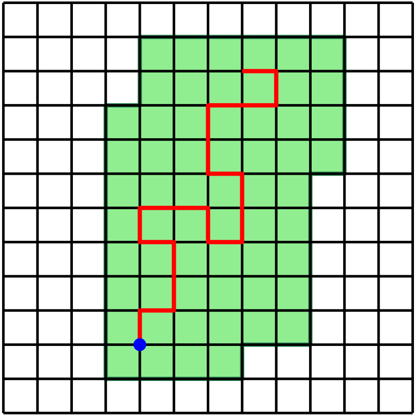

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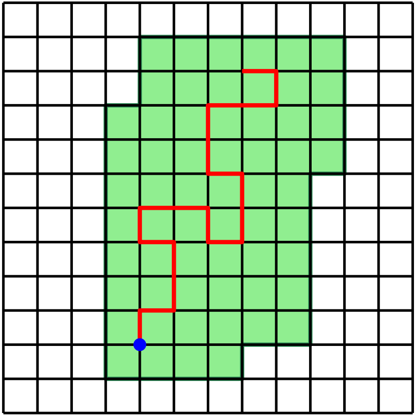

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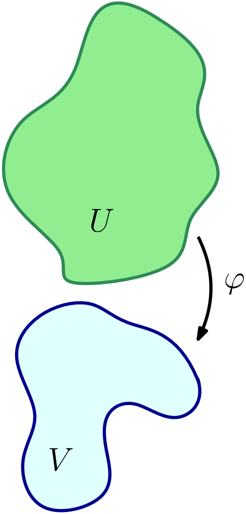

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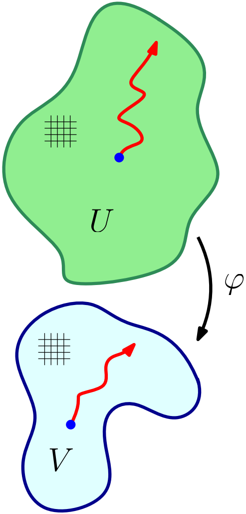

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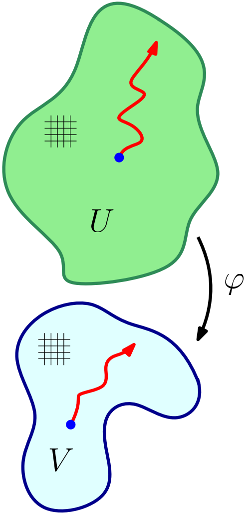
- This talk is about proving a version of this conjecture, but where the underlying graph is a **random planar map**.



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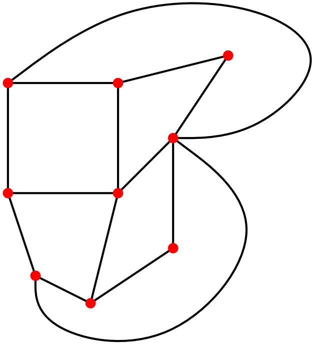
Random planar maps



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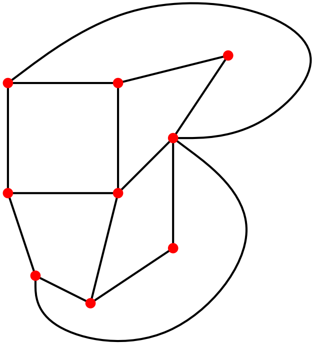

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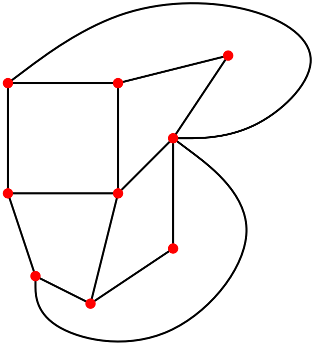

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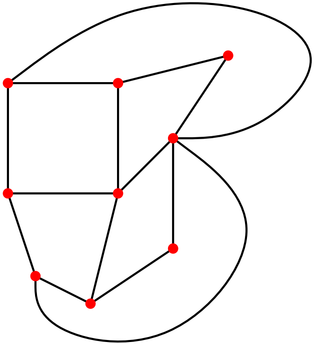

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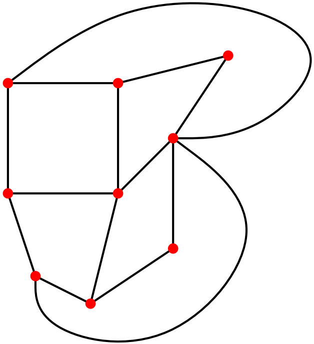

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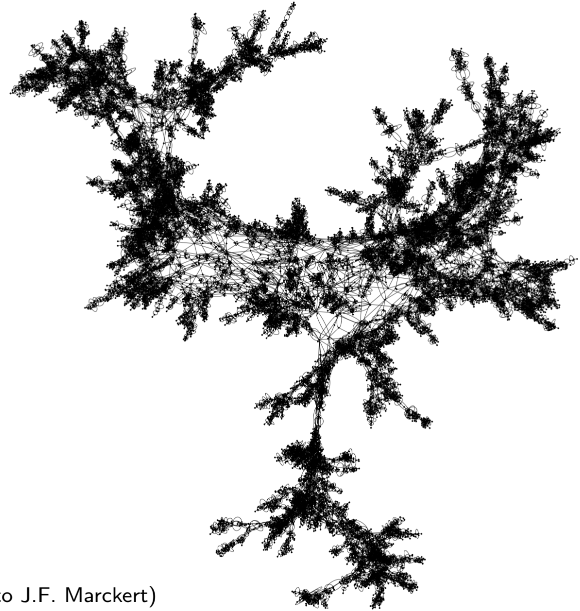
- In this talk, interested in **uniformly random □'s** — **random planar map** (RPM).

Random □ with 25,000 faces



(Simulation due to J.F. Marckert)

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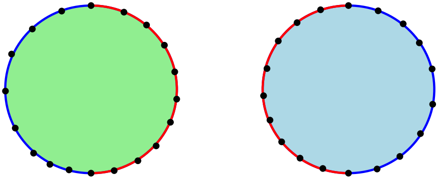

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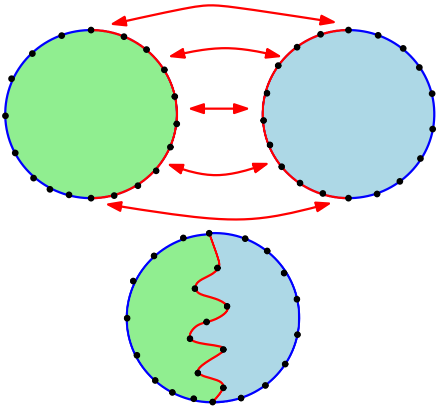

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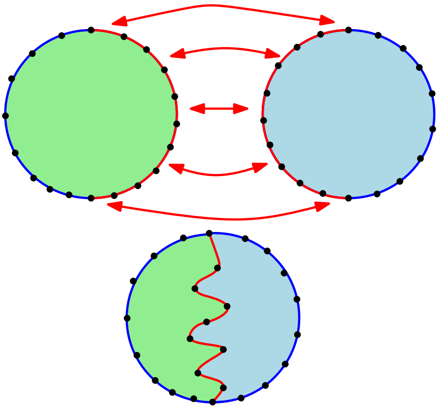

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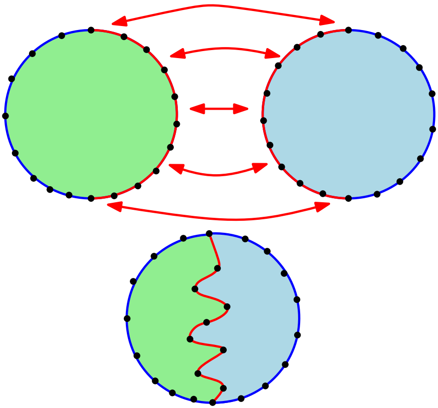

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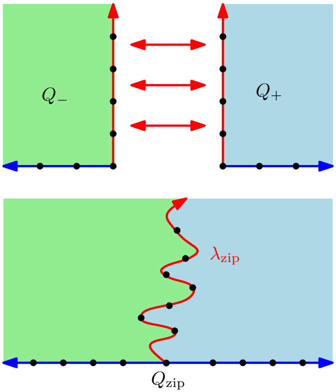

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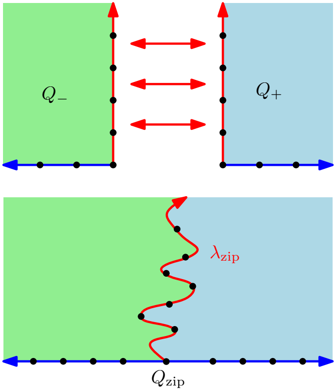
- Glue independent $\mathrm{UIHPQ}_{\mathrm{S}}$'s to get □ of **H** decorated by a simple path. Conditional law of path given □ is a SAW.
- **Goal:** prove scaling limit result for the map/path and identify it with chordal $\mathrm{SLE}_{8/3}$ on $\sqrt{\frac{8}{3}}$-Liouville quantum gravity.



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Comment: For maps with ∂, also have convergence of the boundary path in the uniform topology. The overall topology is the Gromov-Hausdorff-Prokhorov-uniform (GHPU) topology (metric space + measure + path).

Metric gluing

- Metric spaces $M_1 = (X_1, d_1)$, $M_2 = (X_2, d_2)$
- $W = X_1 \sqcup X_2$, $d_\sqcup$ induced natural metric on W, $\sim$ an equivalence relation.
- Set
$$d_{\rm glue}(x,y) = \inf\left\{\sum_{i=1}^n d_\sqcup(a_i, b_i)\right\}$$
where the inf is over all sequences with $a_1 = x$, $b_n = y$, and $b_i \sim a_{i+1}$ for each i.

Then $(W, d_{\rm glue})$ is the **metric gluing** of M_1 and M_2.

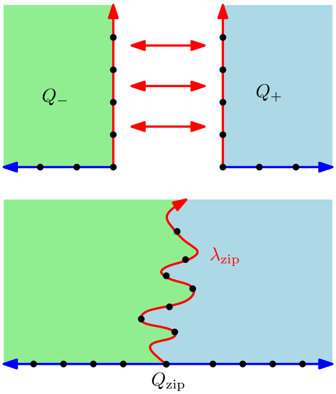
Main example: M_1, M_2 independent instances of the Brownian half-plane identified according to boundary length along their positive boundary rays.

- Metric gluing can be **subtle**
- **Not obvious:** gluing of Brownian half-planes is homeomorphic to $\mathbf{H}$ or that the interface between the two Brownian half-plane instances is a non-trivial curve
- **Worry:** the interface *could even degenerate to a point*

Main scaling limit result

Theorem (Gwynne-M.)

Graph gluing of two independent instances of the UIHPQ$_{\mathrm{S}}$ *converges to the metric gluing of independent Brownian half-plane instances in the GHPU topology. Moreover, the limiting space is homeomorphic to* **H** *and the limiting interface is a non-trivial curve.*



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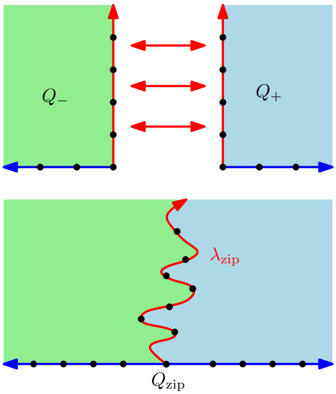

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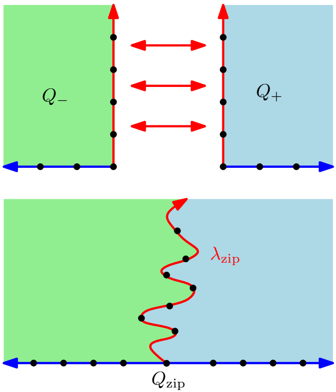

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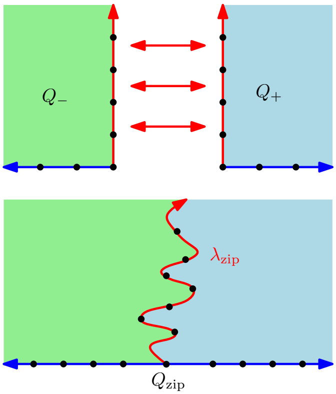

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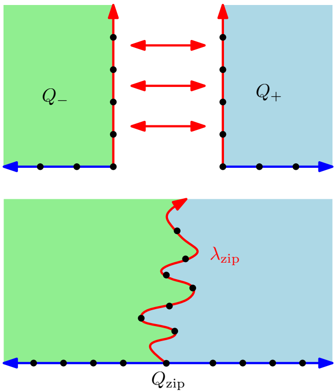

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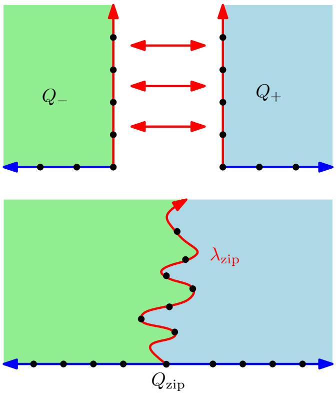

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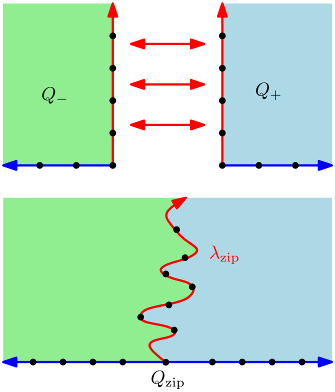

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- **Later:** the limiting space/path pair is isometric to chordal $\mathrm{SLE}_{8/3}$ on $\sqrt{8/3}$-Liouville quantum gravity.

Part II: Liouville quantum gravity

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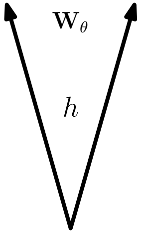
- For other $\gamma \in (0,2)$, γ-LQG arises as the scaling limit of a random planar map decorated with a statistical physics model (peanosphere)

Quantum wedge

- Surfaces with boundary parameterized by **H**. Two special points: origin and ∞.



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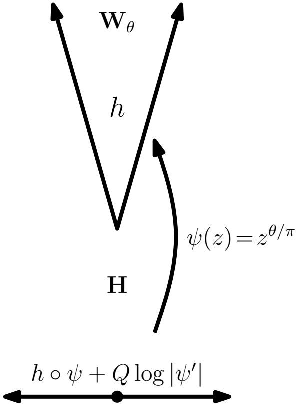

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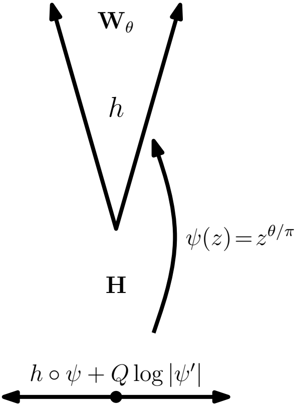

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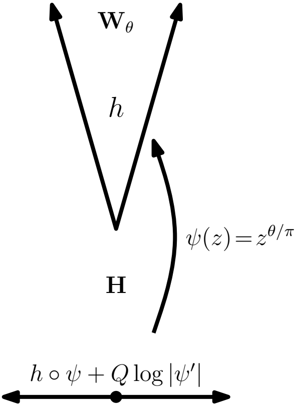

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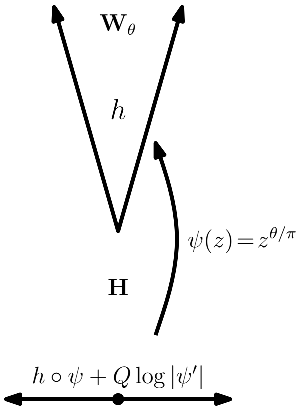

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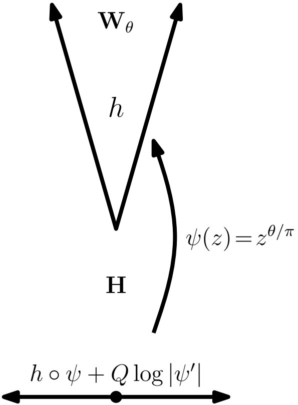
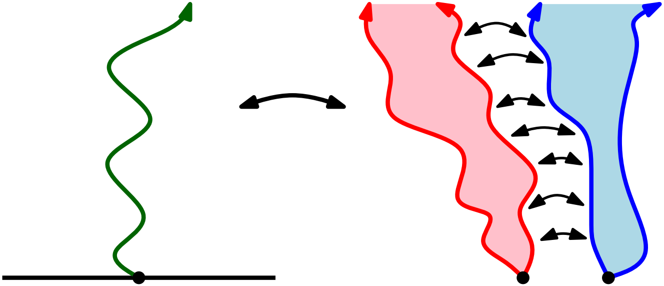
- $\gamma = \sqrt{8/3}$, $\alpha = \gamma$ ($W = 2$), then the quantum wedge is equivalent to the Brownian half-plane.



Cutting and gluing operations



- Cut with an independent chordal SLE curve η or
- Weld together according to boundary length
  - Abstract measurability result: $\mathcal{W}, \eta$ are determined by $\mathcal{W}_1, \mathcal{W}_2$.
  - For $\gamma = \sqrt{8/3}$, not clear that the welding operation is "compatible" with the metric notion of gluing

Metric gluing theorem

Recall: Weight $W = 2$ quantum wedge is equivalent to the Brownian half-plane

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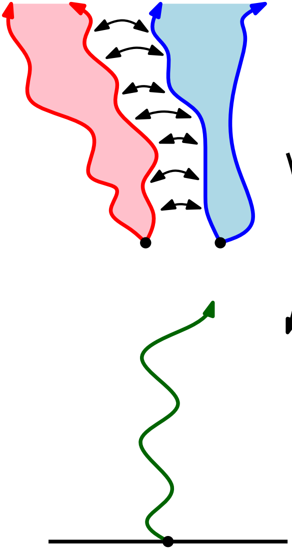

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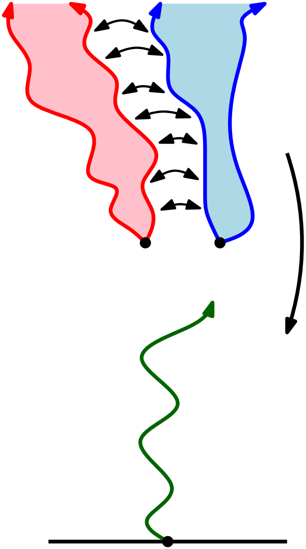
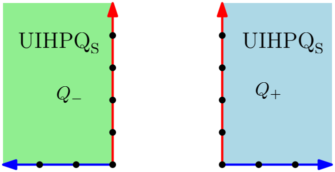
Consequence: if we metrically glue two instances of the Brownian half-plane, the interface between them is exactly a chordal $\mathrm{SLE}_{8/3}$.



Recap



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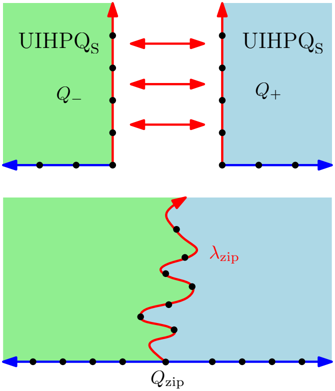
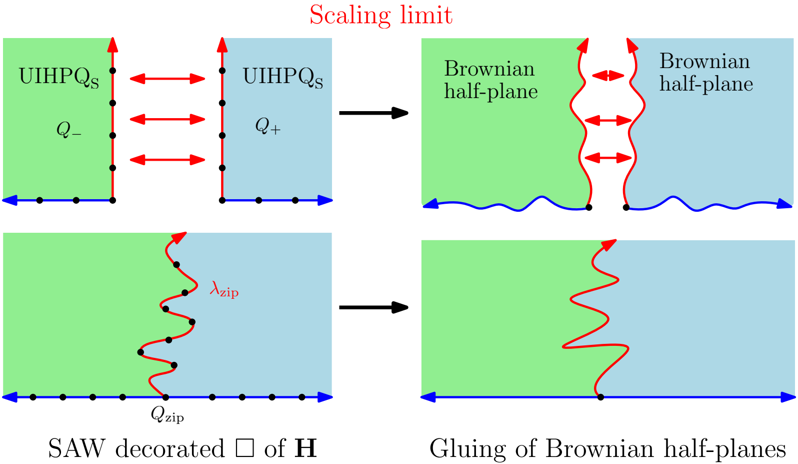


SAW decorated $\square$ of $\mathbf{H}$

Recap


Scaling limit
UIHPQ$_S$
Q_-
UIHPQ$_S$
Q_+
Brownian half-plane
Brownian half-plane
λ_{zip}
Q_{zip}
SAW decorated □ of $\mathbf{H}$
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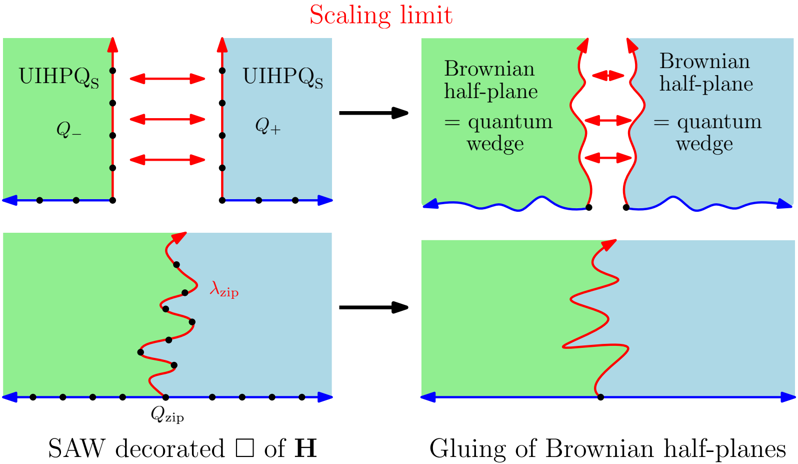

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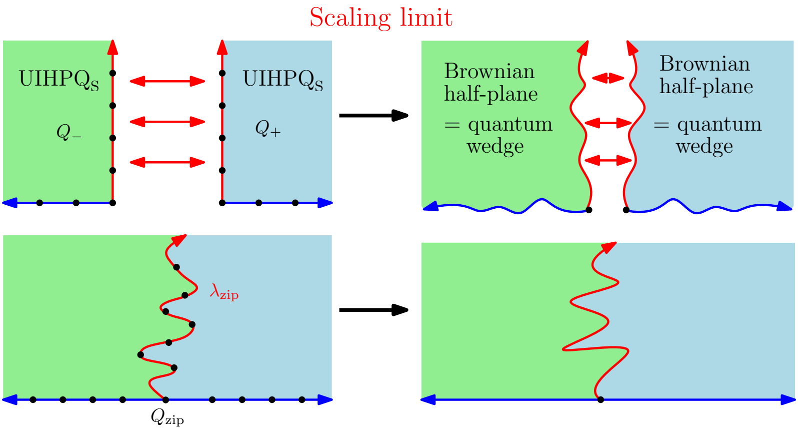

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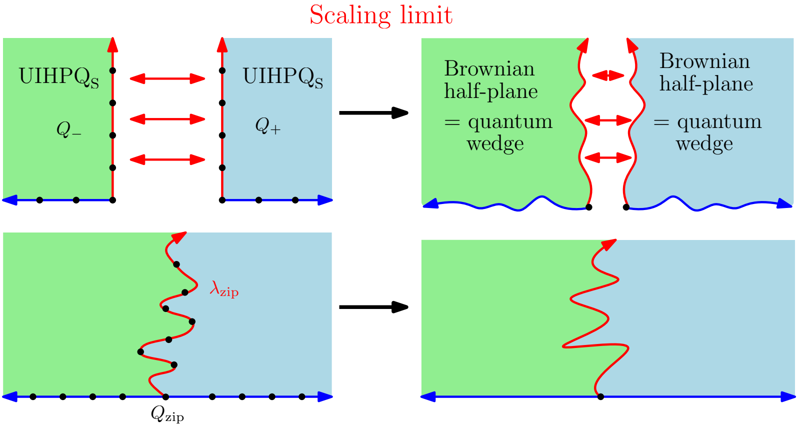
= chordal $\text{SLE}_{8/3}$ on $\sqrt{8/3}$-LQG

Recap



SAW decorated $\square$ of $\mathbf{H}$

Gluing of Brownian half-planes $=$ chordal $\mathrm{SLE}_{8/3}$ on $\sqrt{8/3}$-LQG

Consequence: SAW on random $\square$'s converges to $\mathrm{SLE}_{8/3}$ on $\sqrt{8/3}$-LQG

Part III: Proof ideas

Overview

Will focus on the discrete to continuum convergence statement

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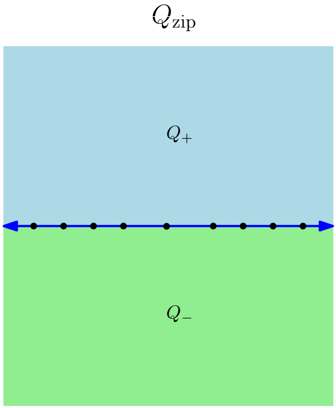
- **Goal:** show that the discrete graph gluing of two independent UIHPQ$_S$'s Q_-, Q_+ converges in the limit to the metric gluing of independent Brownian half-planes



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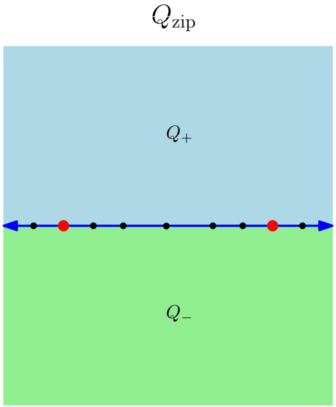

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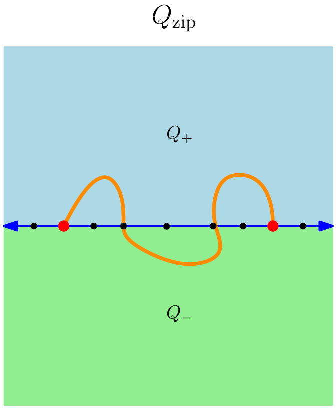

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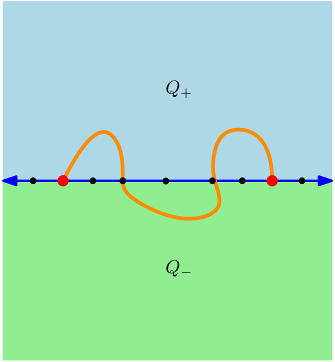
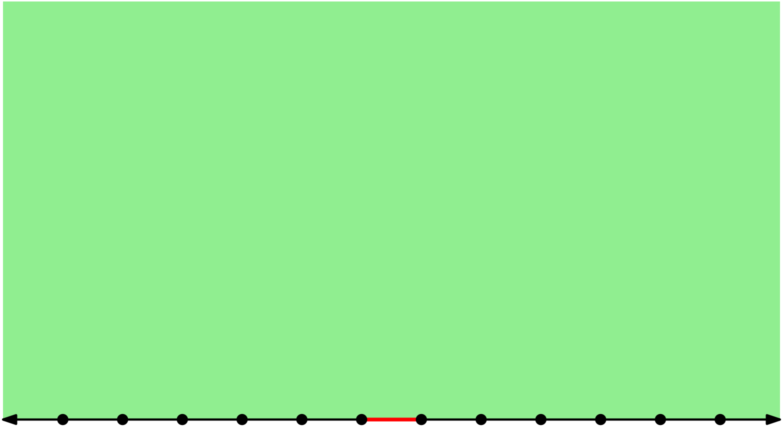
- **Challenge:** Understand the structure of the metric along the interface in a precise way

Q_{zip}



Peeling the UIHPQ$_S$



UIHPQ$_S$ with marked edge in **red**.

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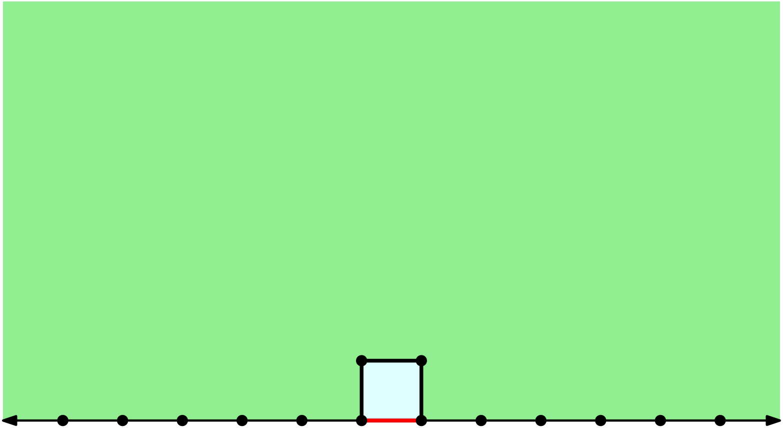

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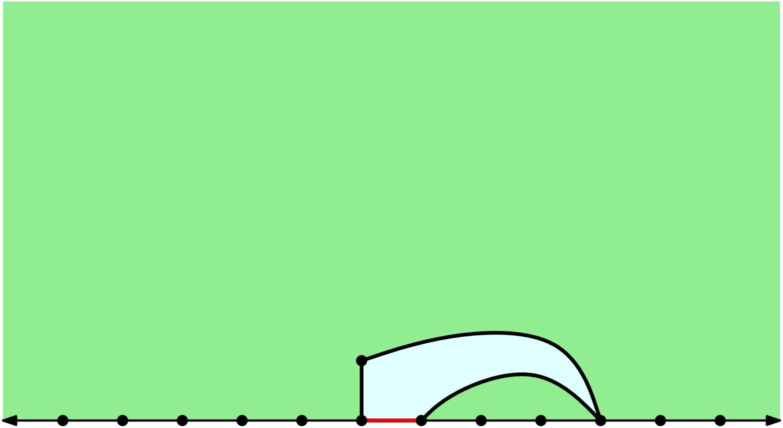

UIHPQ$_S$ with marked edge in **red**. Reveal the $\square$ adjacent to the marked edge.

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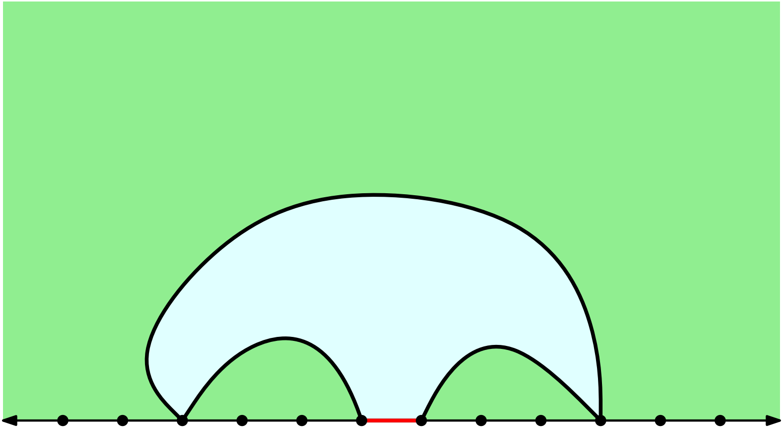

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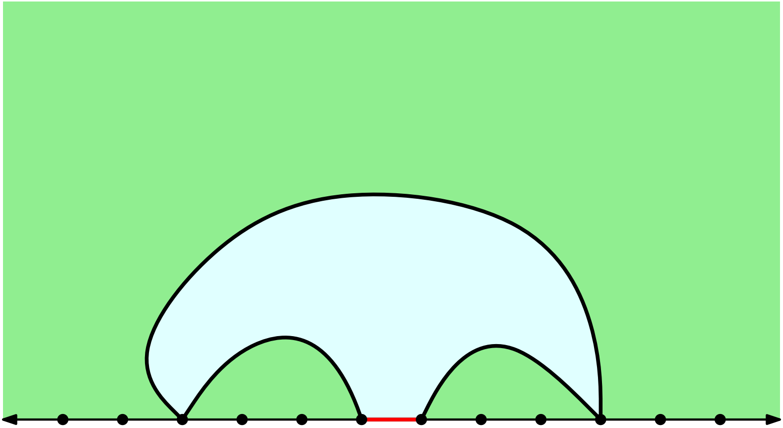

UIHPQ$_S$ with marked edge in **red**. Reveal the $\square$ adjacent to the marked edge. Exact formulas for the probability of each possibility.

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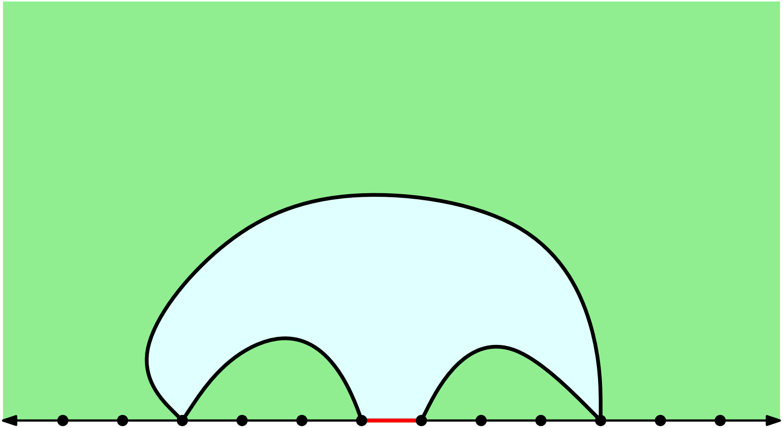

UIHPQ$_S$ with marked edge in **red**. Reveal the $\square$ adjacent to the marked edge. Exact formulas for the probability of each possibility. Unexplored region is a UIHPQ$_S$.

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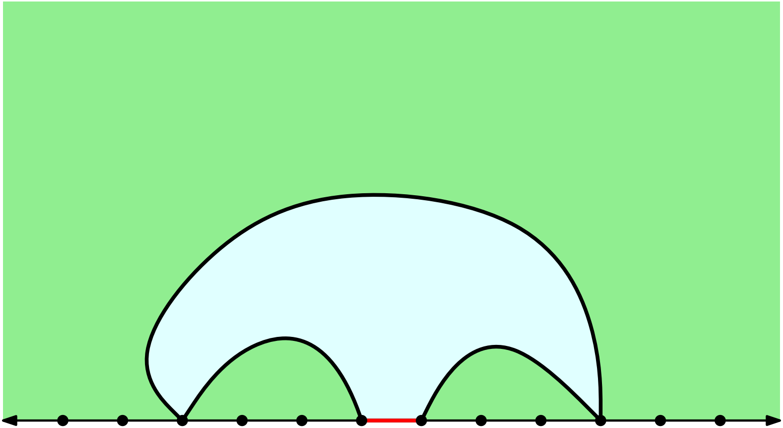

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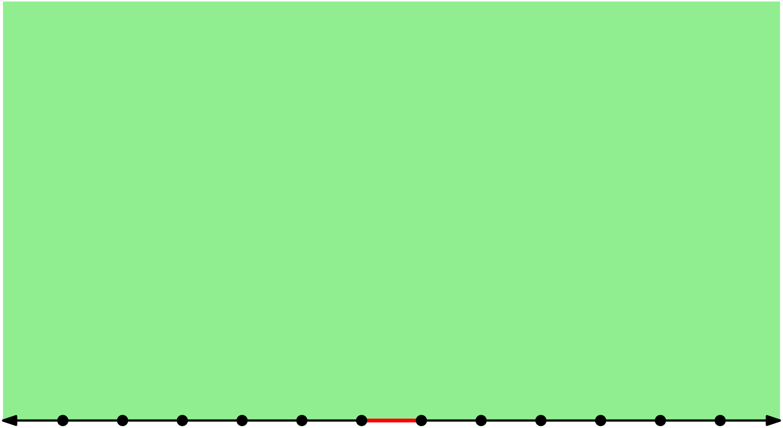
Peeling the UIHPQ$_S$



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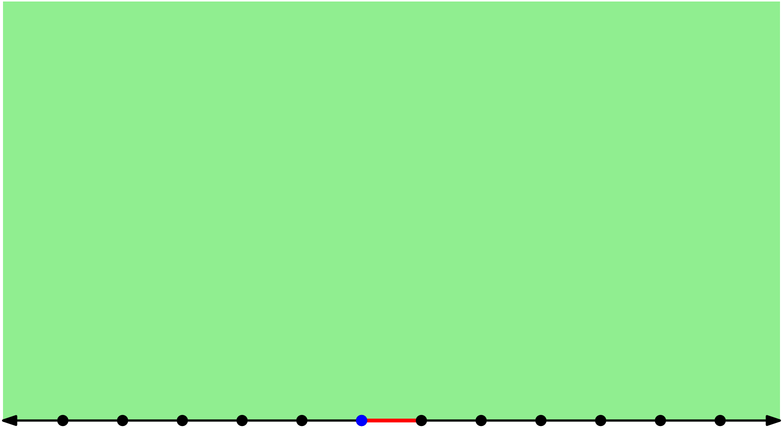

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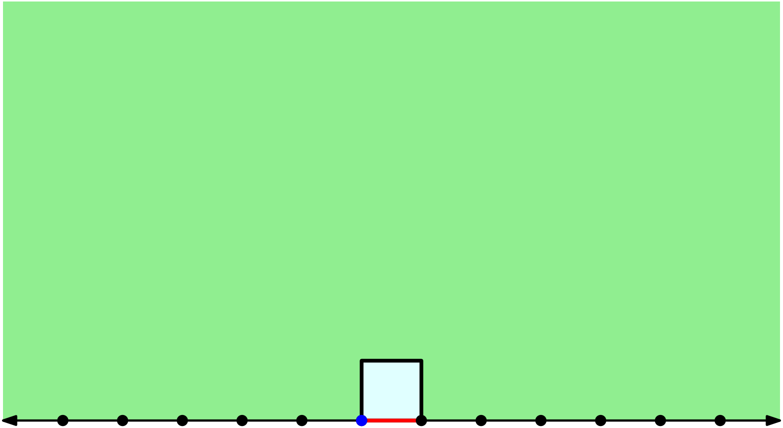

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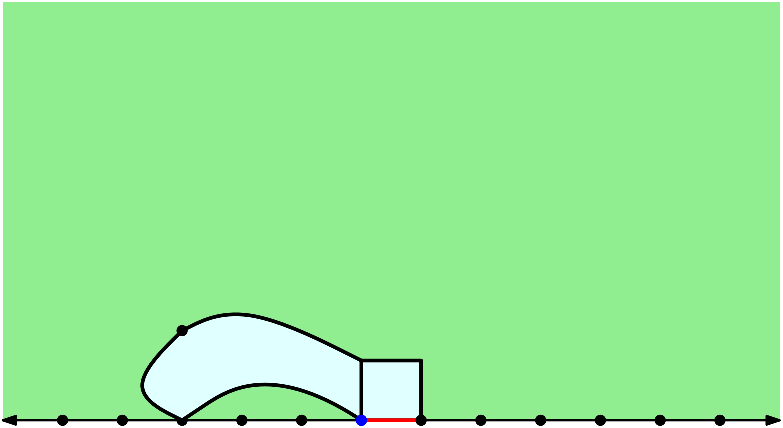

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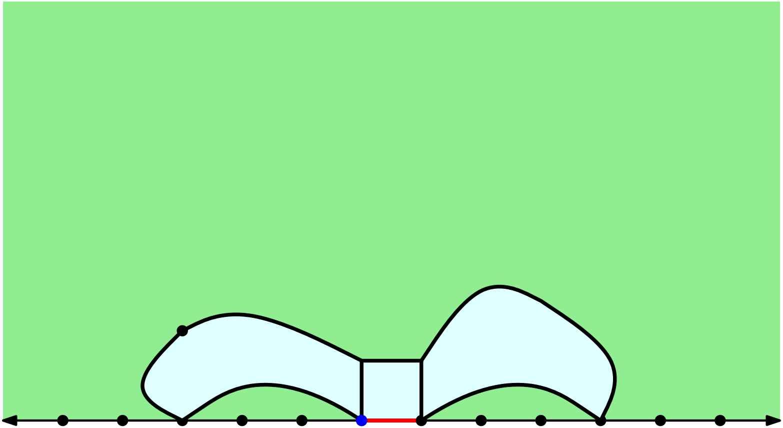

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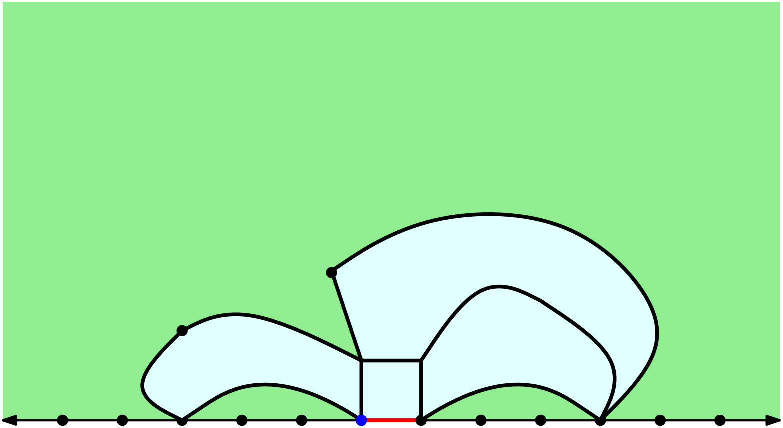

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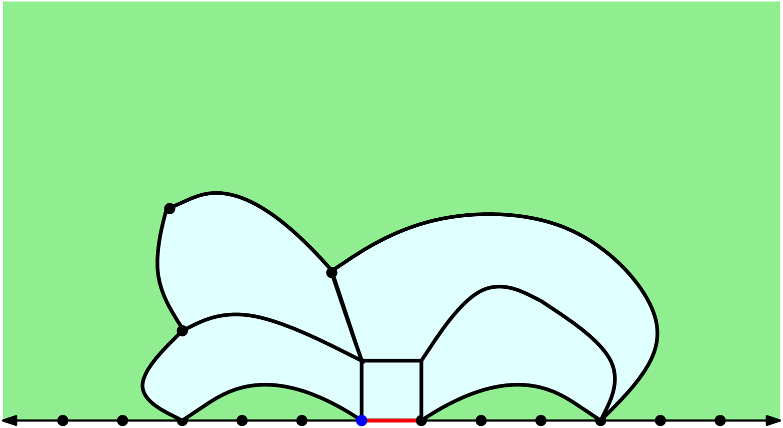
Peeling the UIHPQ_S



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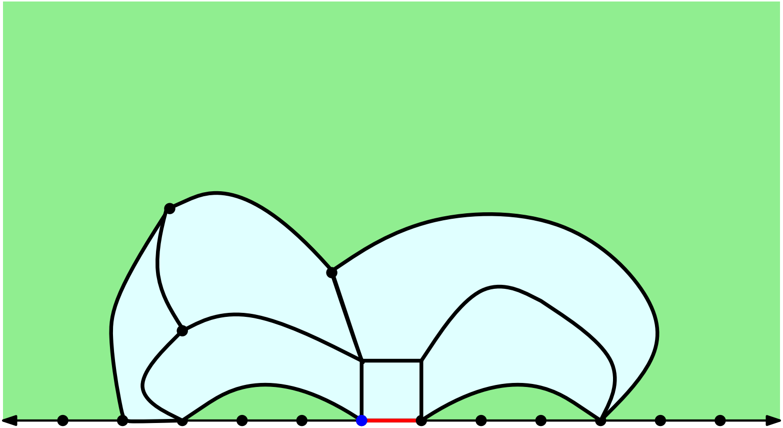

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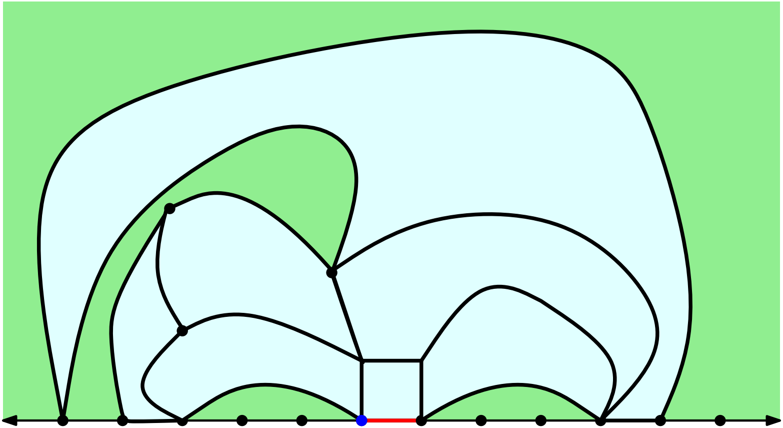

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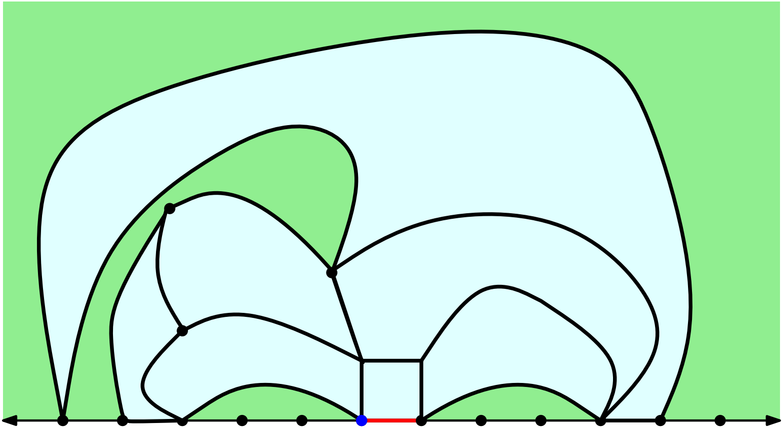
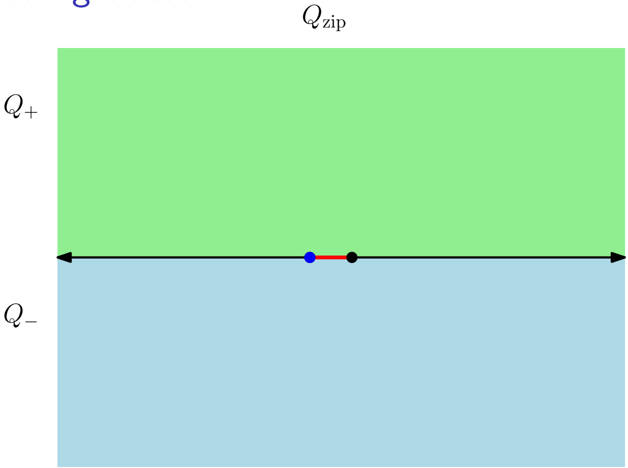
Peeling the UIHPQ$_S$



UIHPQ$_S$ with marked edge in **red**. Reveal the □ adjacent to the marked edge. Exact formulas for the probability of each possibility. Unexplored region is a UIHPQ$_S$. Probability disconnect k_1 edges on the left and k_2 edges on the right is $\cong k_1^{-5/2} k_2^{-5/2}$. Metric ball exploration targeted at ∞ via peeling. Can control precisely the ∂-length.

Glued peeling cluster



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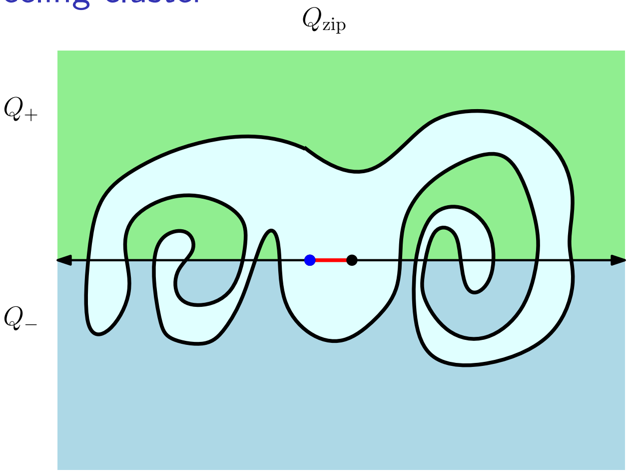

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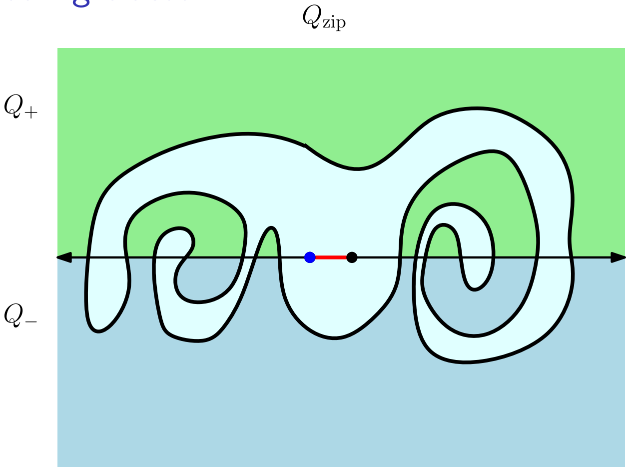

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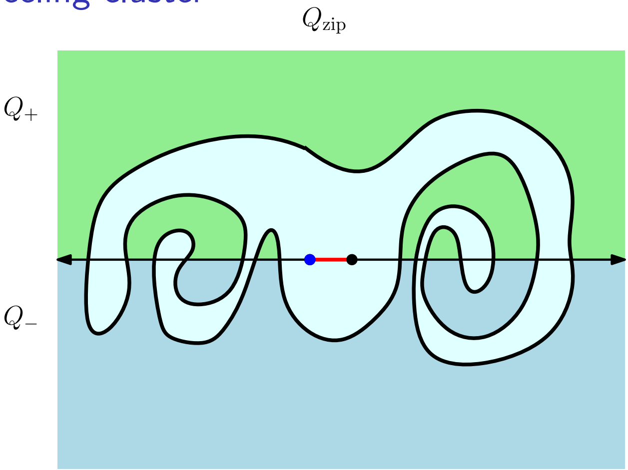

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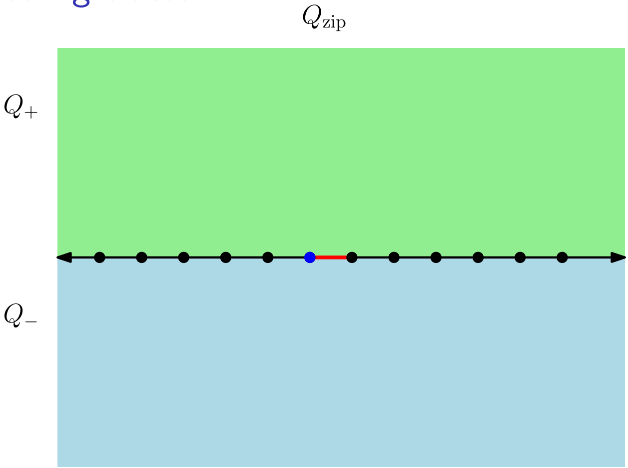

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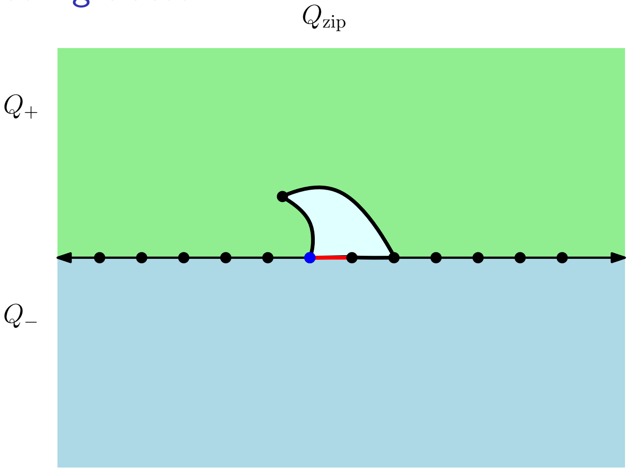

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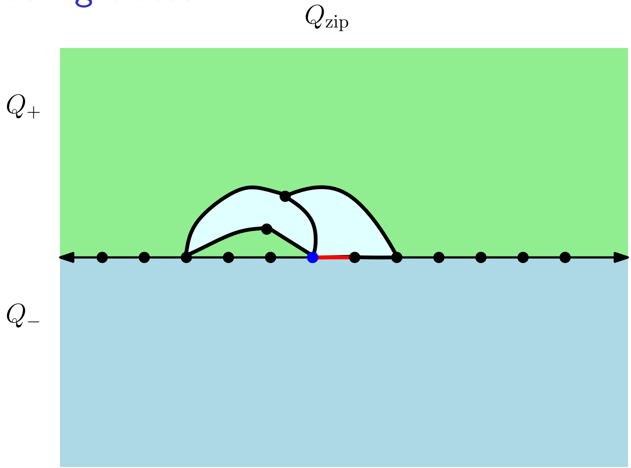

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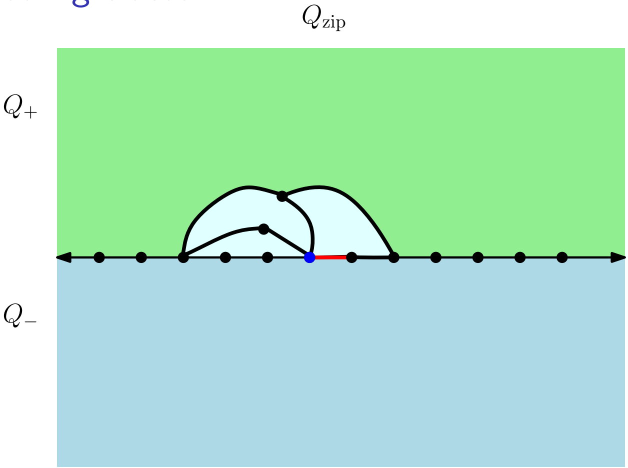

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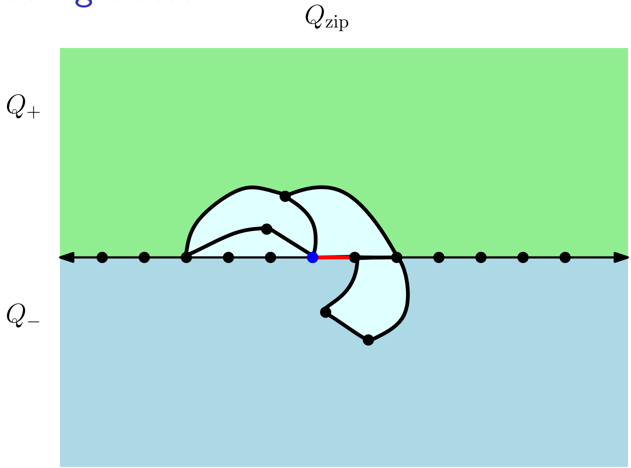

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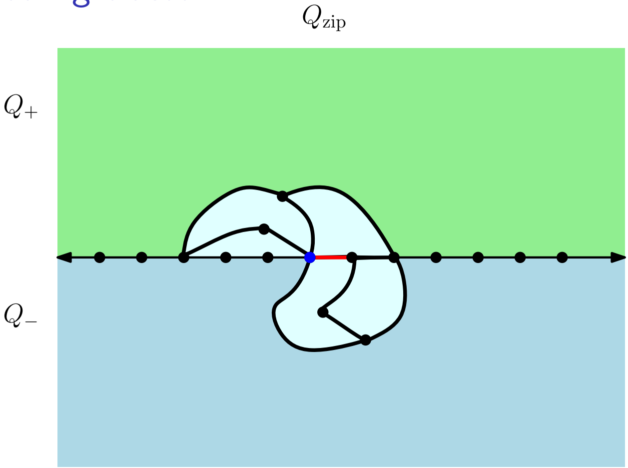

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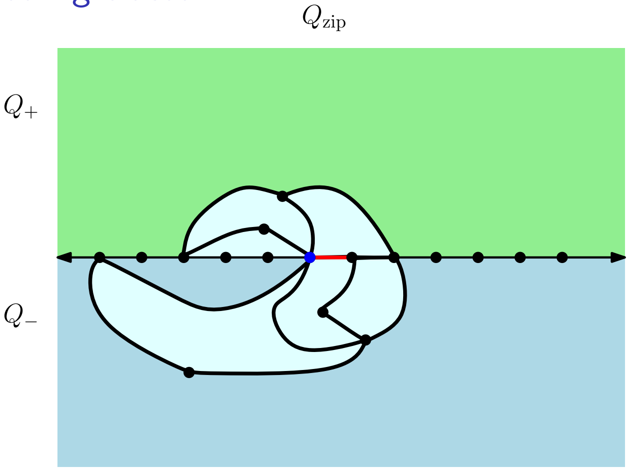

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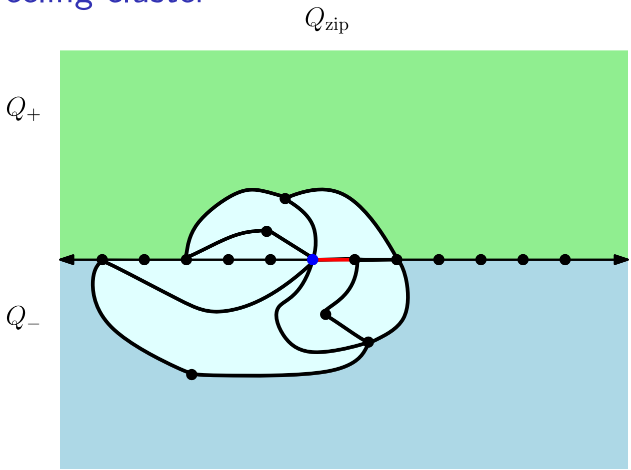

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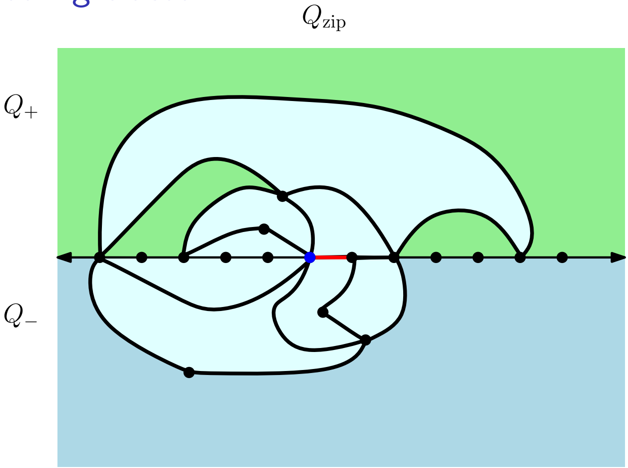

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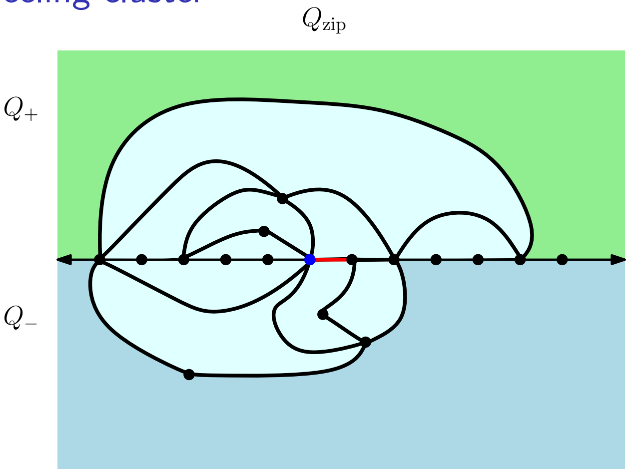

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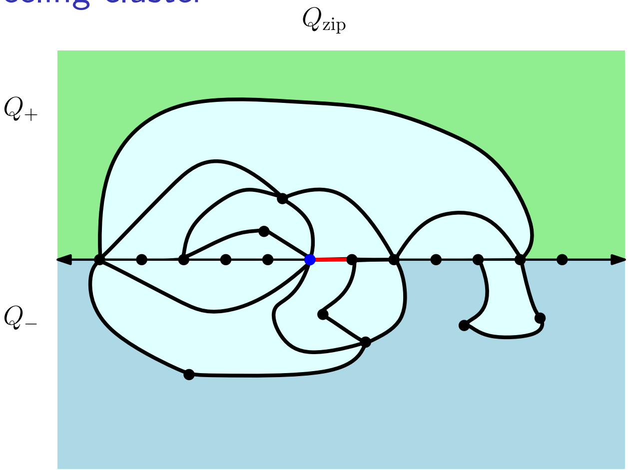

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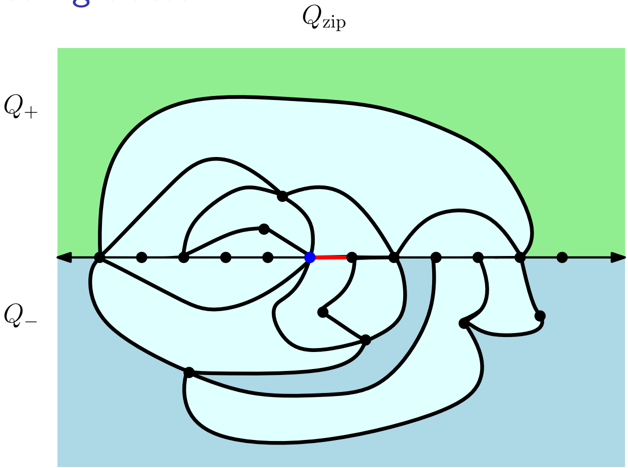

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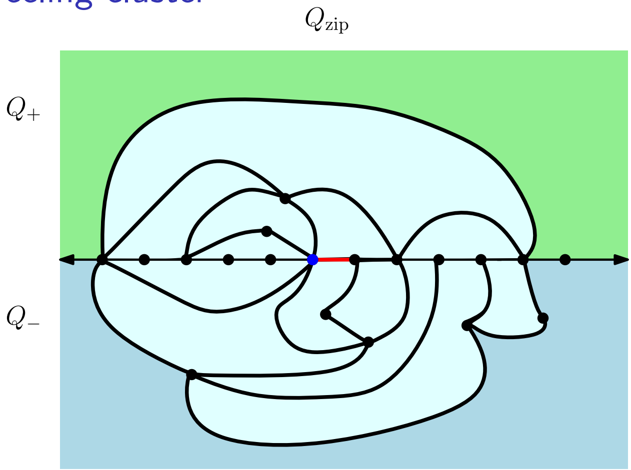

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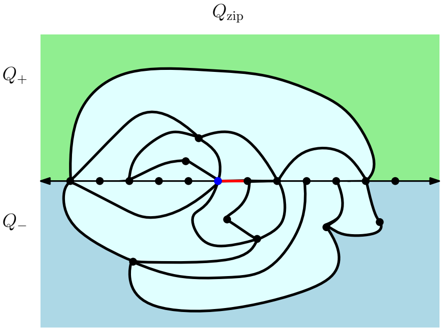
Controlling the glued peeling cluster

- **Recall:** Glued peeling cluster $\geq$ metric ball



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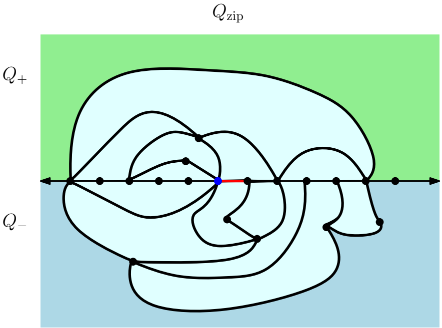
- **Recall:** Glued peeling cluster $\geq$ metric ball
- ∂-length and area harder to control due to to the upward jumps in boundary length



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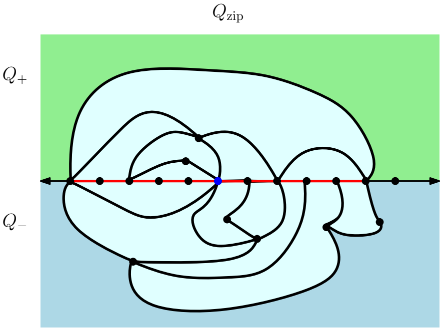

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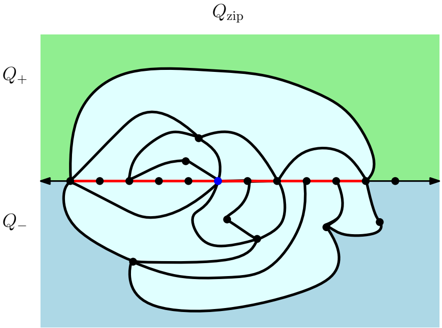

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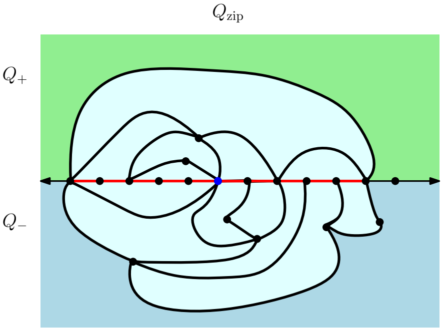
- **Corollary:** (Gwynne, M.) The interface is non-degenerate in the subsequential limit.



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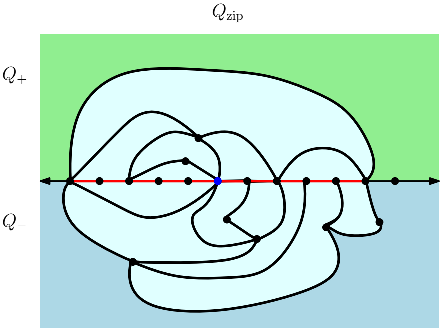
- **Proof idea:** Recursive moment bounds for ∂-length and edges cut off from ∞.



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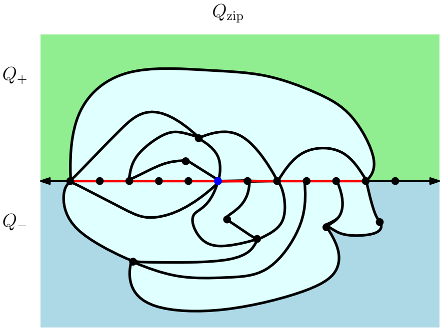
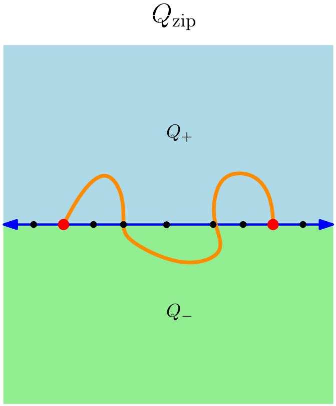


Caraceni-Curien also studied SAWs on random $\square$'s and used the glued peeling cluster. Controlled the $p = 1$ moment of the set of edges cut off from ∞.

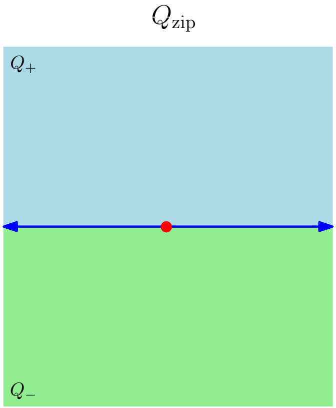
Finishing the proof

Recall: goal is to show that a geodesic connecting ∂ points of ∂ distance $n^{1/2}$ from each other can be approximated by a path which crosses the interface at most a finite number of times (not growing with n).



Finishing the proof

- Consider glued peeling clusters at dyadic scales



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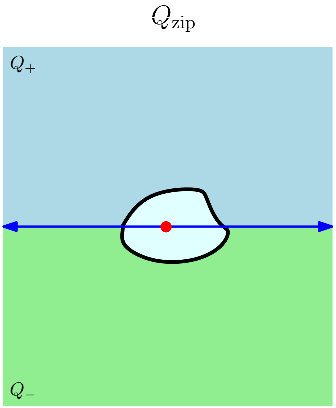

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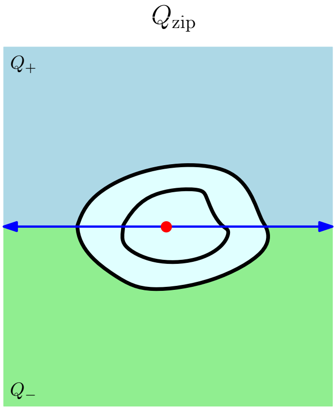

Finishing the proof

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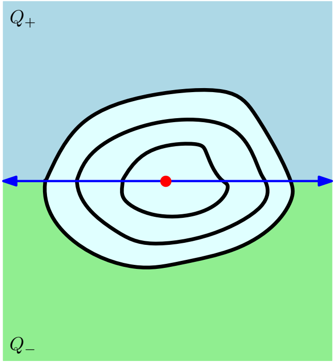
$\mathcal{Q}_{\text{zip}}$



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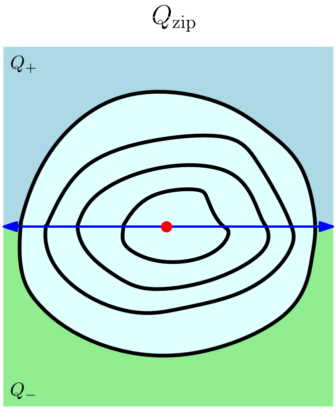
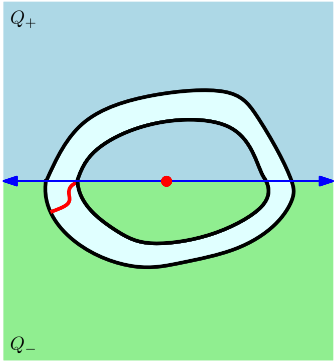


Finishing the proof

- Consider glued peeling clusters at dyadic scales
- Call a scale K-**good** ($K > 1$) if the Q_{zip} distance between any point on the inner and any point on the outer ∂ is at least $1/K$ times the length of a path which crosses the interface at most once.

Q_{zip}



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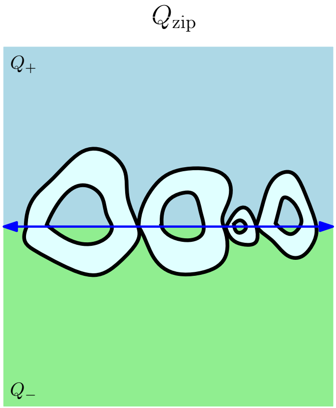

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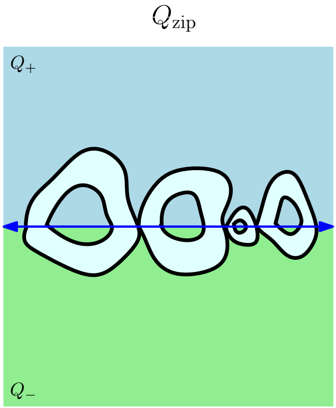

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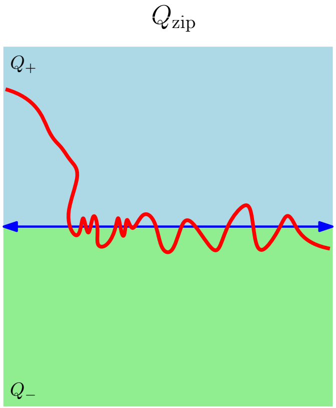

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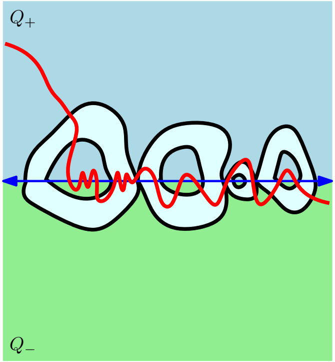
$Q_{\rm zip}$



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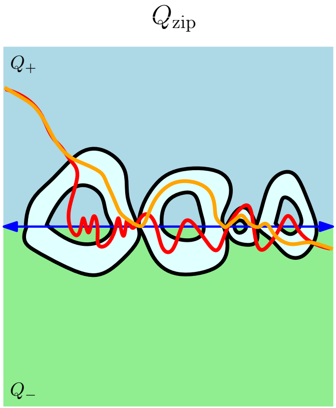

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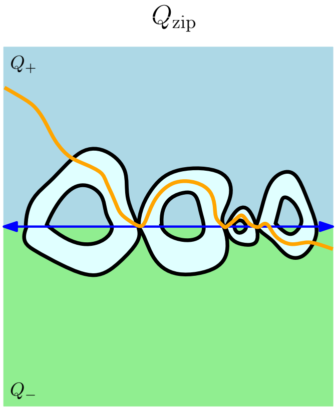

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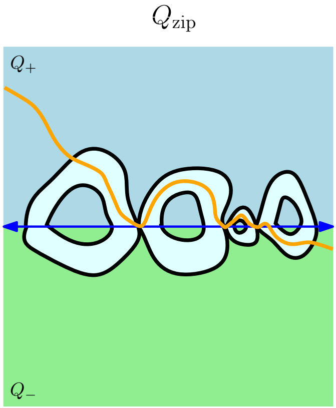

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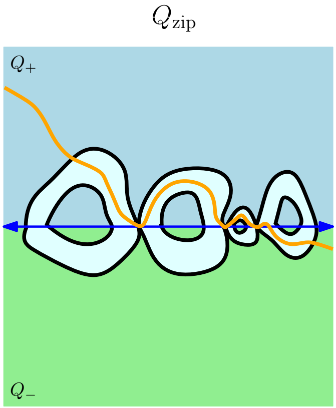

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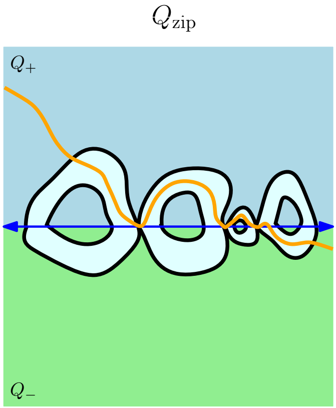
- Call a scale K-**good** ($K > 1$) if the $Q_{\rm zip}$ distance between any point on the inner and any point on the outer ∂ is at least $1/K$ times the length of a path which crosses the interface at most once.
- **Theorem** (Gwynne, M.) Choosing $K > 1$ large enough, we can cover the interface by K-good annuli with high probability.
- **Consequence:** The subsequentially limiting metric of $Q_{\rm zip}$ is bi-Lipschitz equivalent to the metric gluing of Brownian half-planes.
- **Theorem** (Gwynne, M.) There exists $\alpha \in (0,1)$ such that the subsequentially limiting geodesics a.s. spend at most α-fraction of their time in the interface.
- **Consequence:** The subsequentially limiting metric is equivalent to the metric gluing of Brownian half-planes.



Remark: arguments are delicate as the interface has $n^{1/2}$ edges while the geodesic has $n^{1/4}$.

Thanks!