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Liouville quantum gravity and the Brownian map

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Cambridge and MIT

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Overview

Part I: Picking surfaces at random

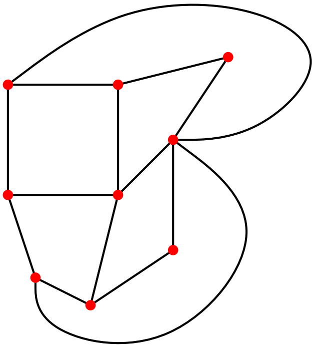
1. Discrete: random planar maps
2. Continuum: Liouville quantum gravity (LQG)
3. Relationship

Part II: The $\mathrm{QLE}(8/3, 0)$ metric on $\sqrt{8/3}$-LQG

1. First passage percolation on random planar maps
2. First passage percolation on $\sqrt{8/3}$-LQG: $\mathrm{QLE}(8/3, 0)$

Part I: Picking surfaces at random

Random planar maps



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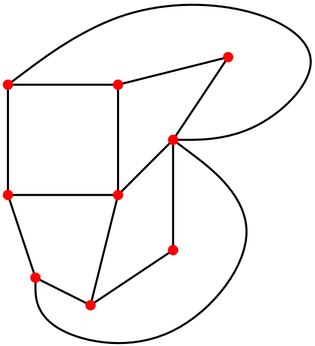

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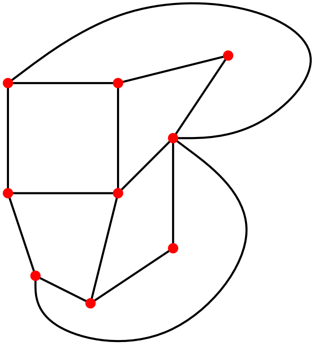

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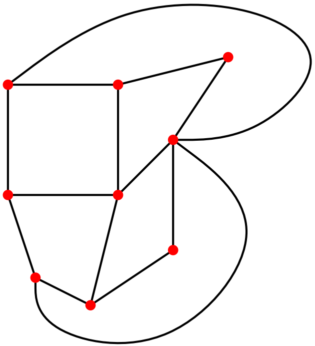

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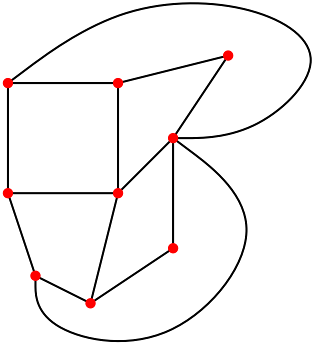

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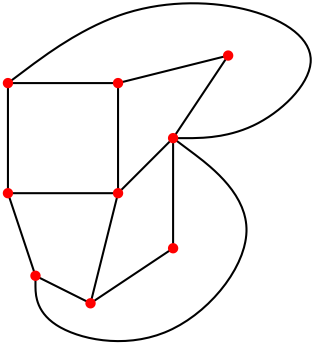

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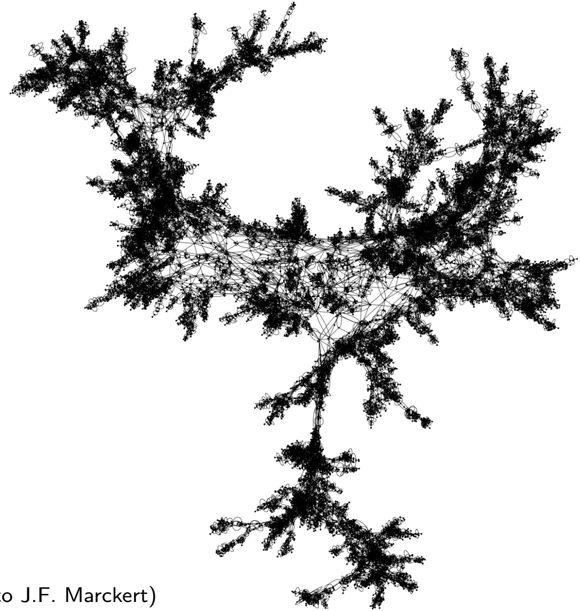
- First studied by Tutte in 1960s while working on the four color theorem
    - **Combinatorics**: enumeration formulas
    - **Physics**: statistical physics models: percolation, Ising, UST ...
    - **Probability**: "uniformly random surface," Brownian surface

Random quadrangulation with 25,000 faces



(Simulation due to J.F. Marckert)

Structure of large random planar maps

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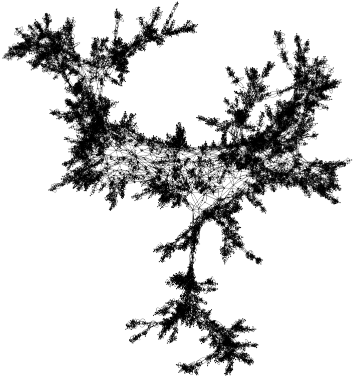

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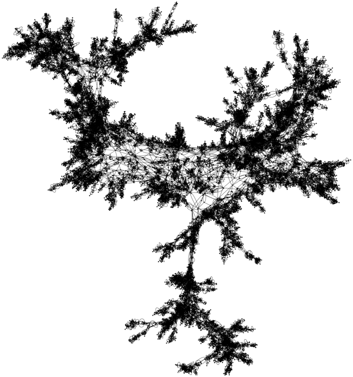

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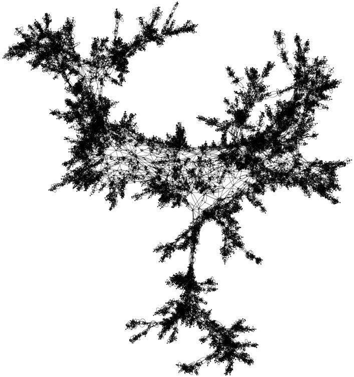

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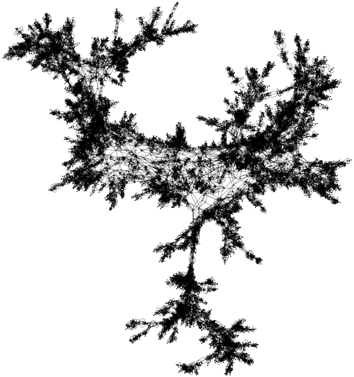

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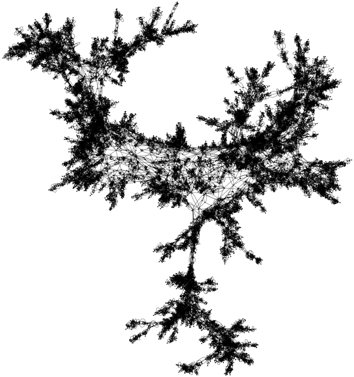

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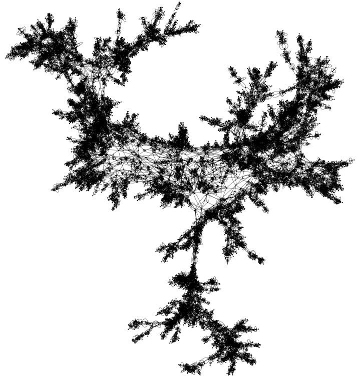

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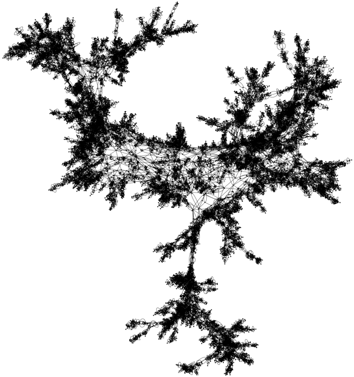

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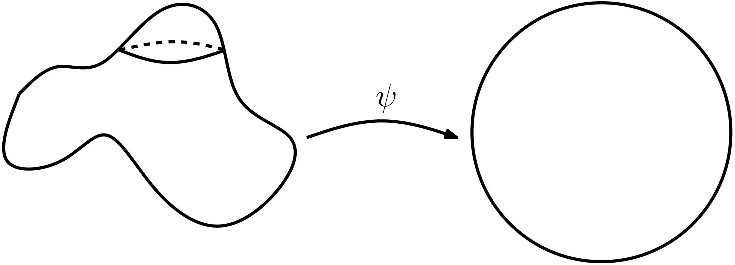
Brownian map also described in terms of trees (CRT)

(Markert-Mokkadem)

Picking a surface at random in the continuum

Uniformization theorem: every Riemannian surface homeomorphic to the unit disk **D** can be conformally mapped to the disk.



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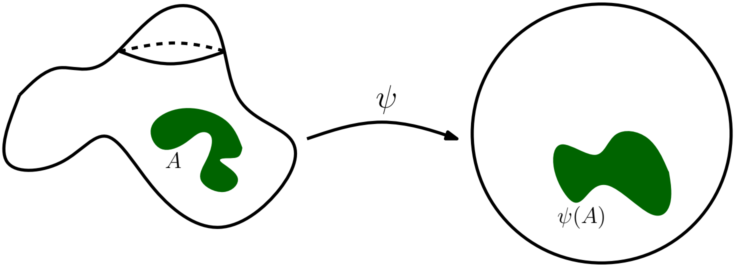

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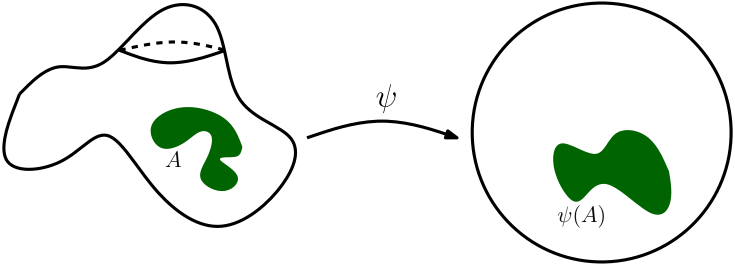

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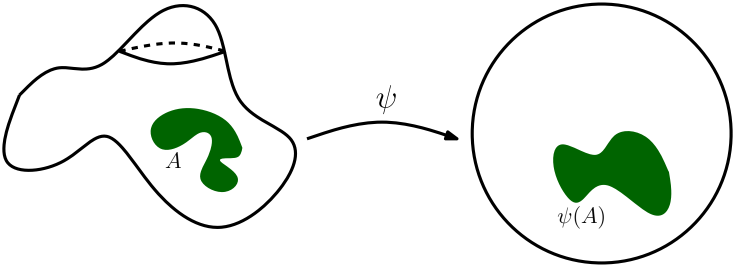

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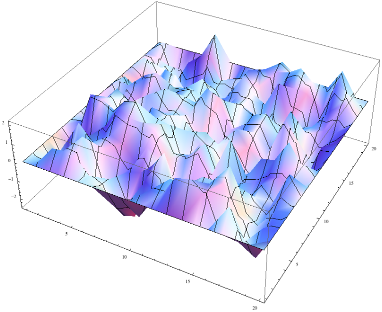
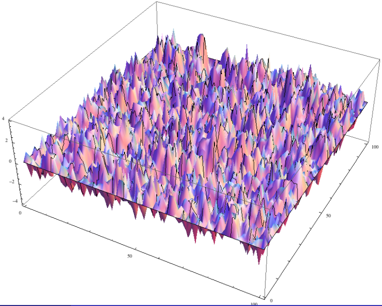
Question: Which measure on ρ? If we want our surface to be a perturbation of a flat metric, natural to choose ρ as the canonical perturbation of a harmonic function.

The Gaussian free field

- The **discrete Gaussian free field** (DGFF) is a **Gaussian random surface** model.





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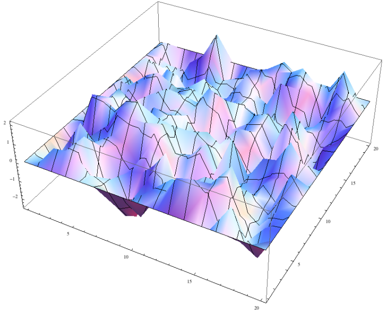
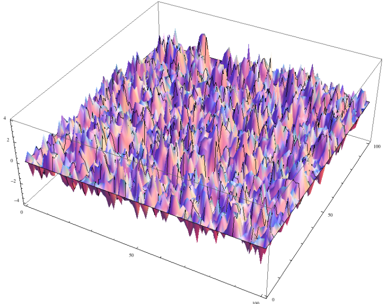
- Measure on functions $h\colon D \to \mathbf{R}$ for $D \subseteq \mathbf{Z}^2$ and $h|_{\partial D} = \psi$ with density respect to Lebesgue measure on $\mathbf{R}^{|D|}$:

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{2} \sum_{x \sim y} (h(x) - h(y))^2 \right)$$





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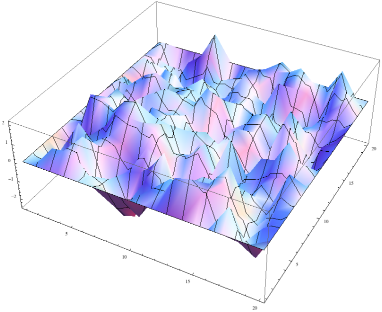

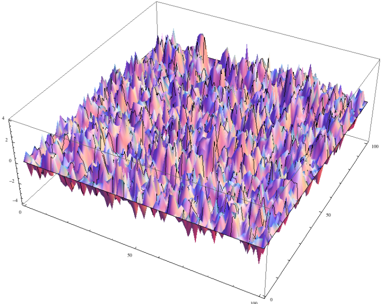

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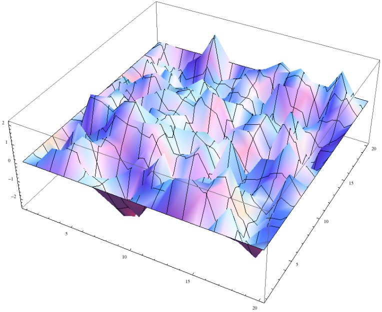

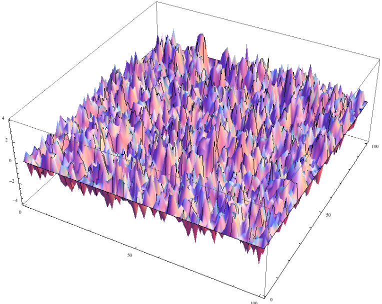

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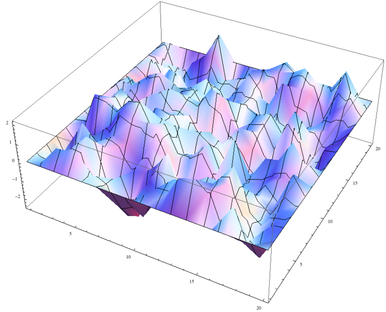
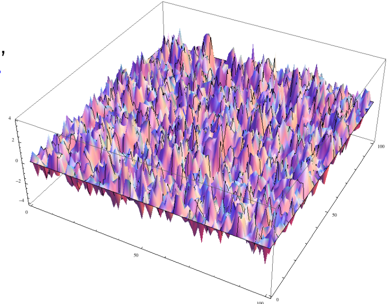
- Continuum GFF not a function — only a generalized function





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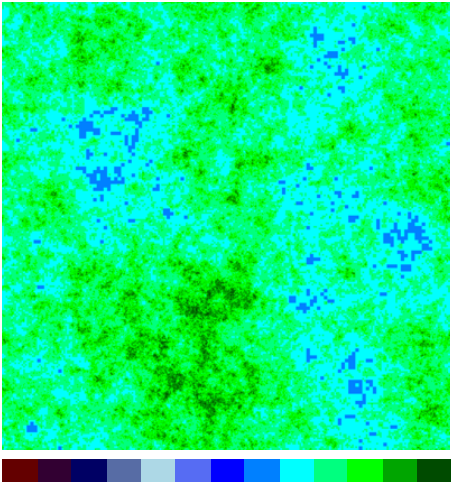

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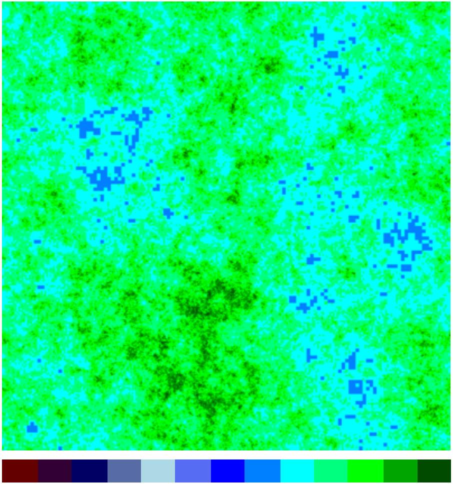

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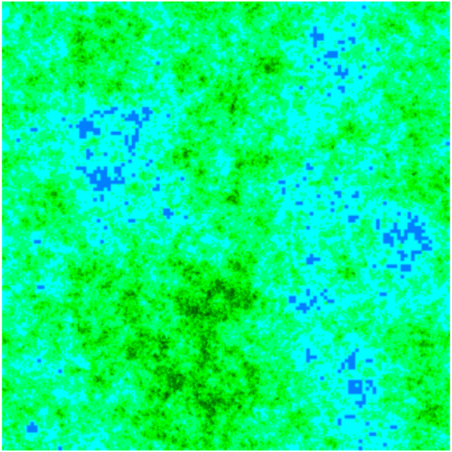

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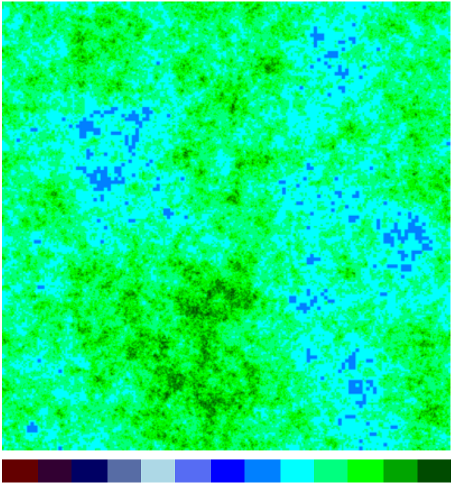

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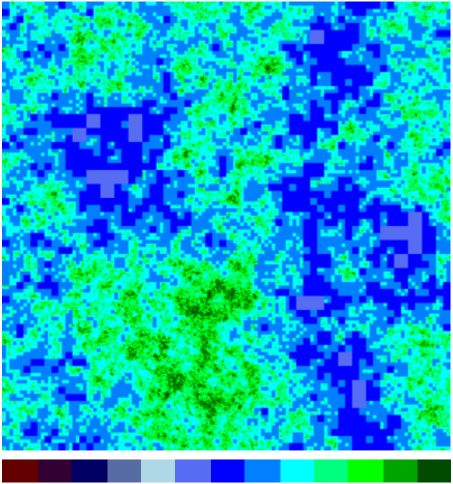

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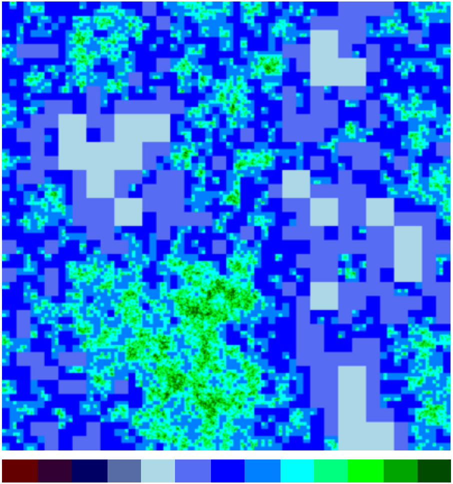

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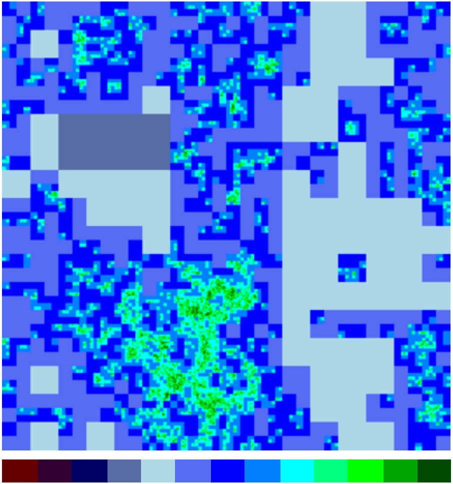

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This talk is about endowing each of these objects with the *other's* structure and showing they are equivalent.

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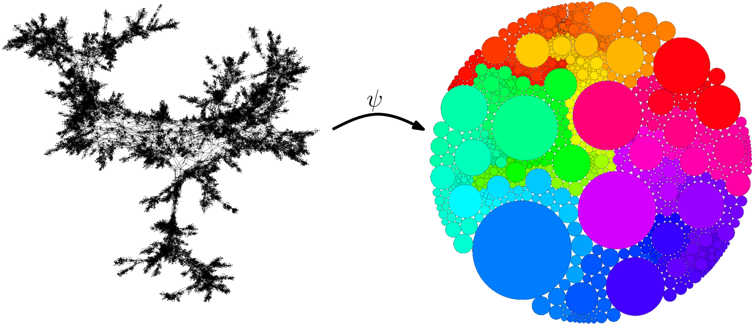

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Canonical embedding of TBM into $\mathbf{S}^2$

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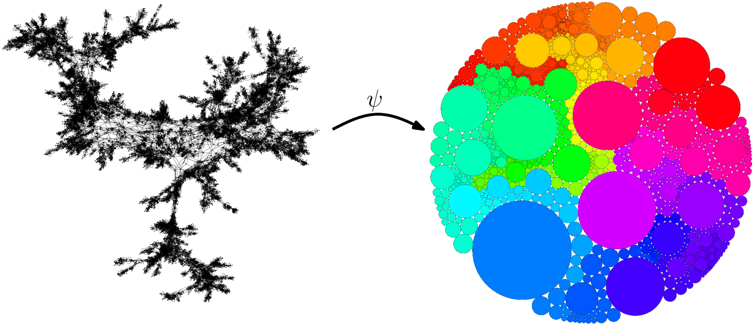

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Main result

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Suppose that (M, d, μ) is an instance of TBM. Then there exists a Hölder homeomorphism $\varphi\colon (M, d) \to \mathbf{S}^2$ such that the pushforward of μ by φ has the law of a $\sqrt{8/3}$-LQG sphere $(\mathbf{S}^2, h)$.

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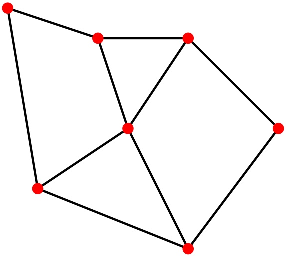
5. Metric construction is for the $\sqrt{8/3}$-LQG sphere. By absolute continuity, can construct a metric on any $\sqrt{8/3}$-LQG surface.

Part II:

Construction of the metric on $\sqrt{8/3}$-LQG

Detour: first passage percolation (FPP)

- Associate with a graph (V, E) i.i.d. $\exp(1)$ edge weights



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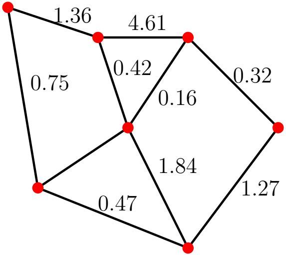

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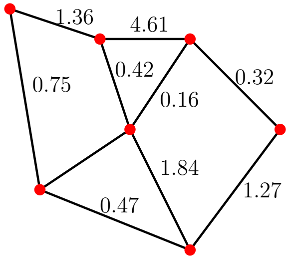

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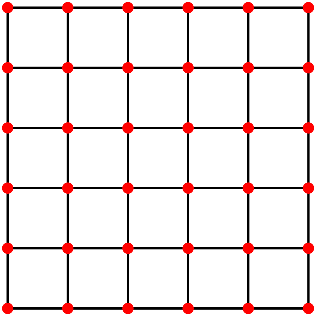

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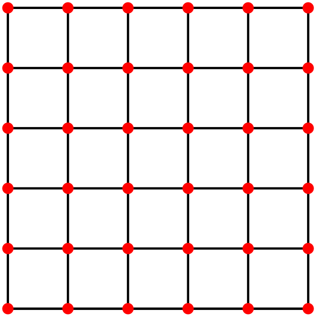

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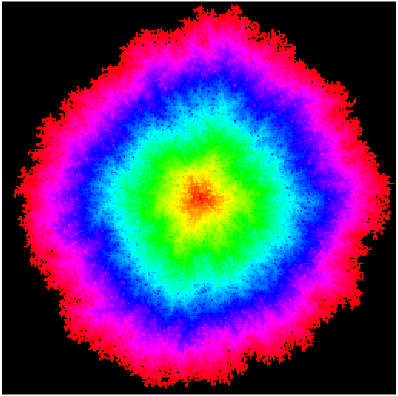

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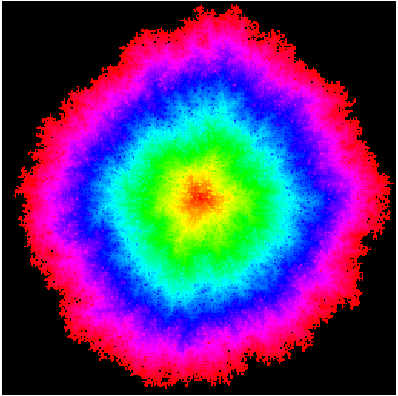

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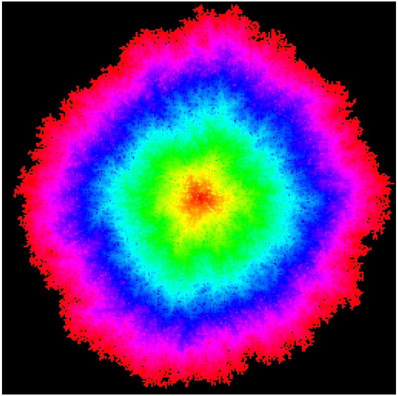

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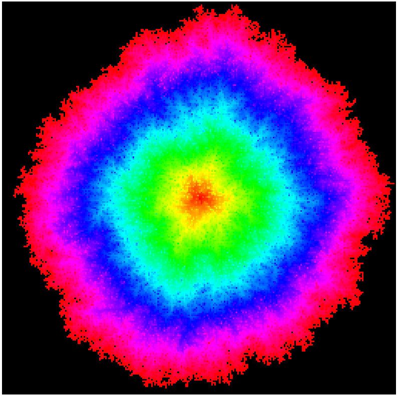

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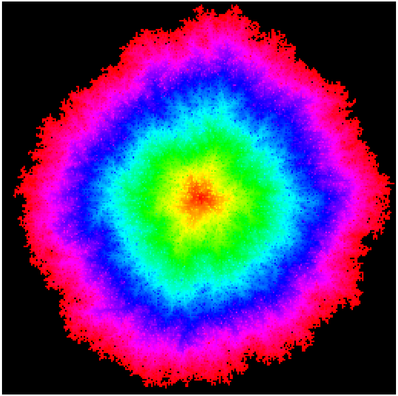

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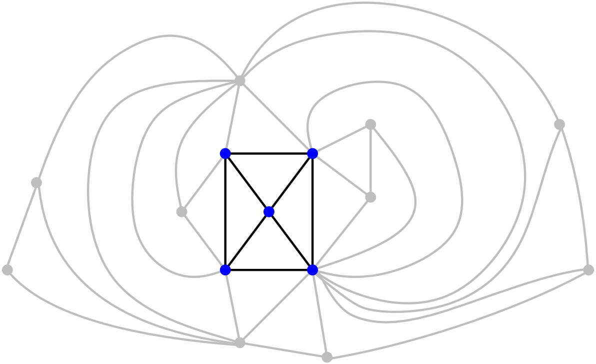

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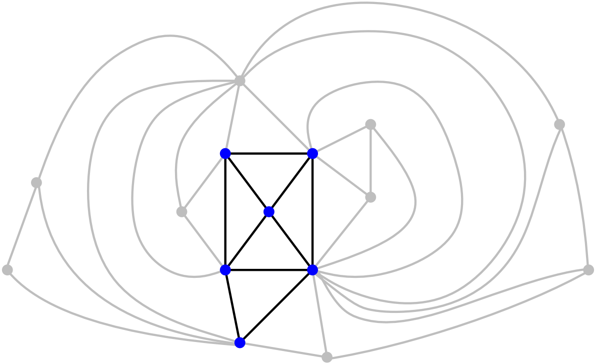

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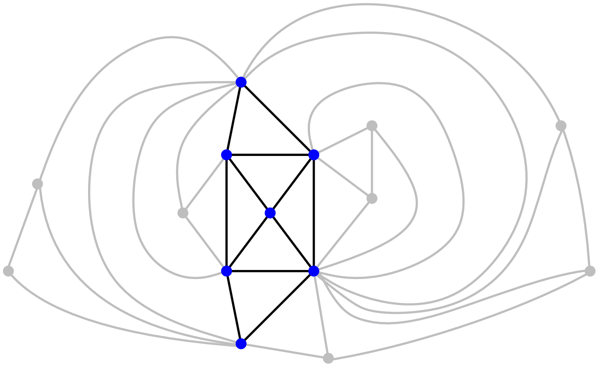

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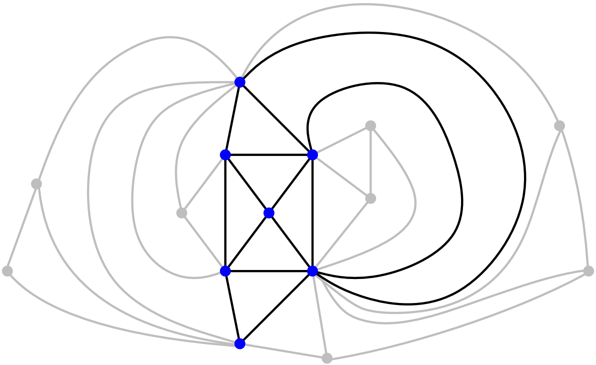

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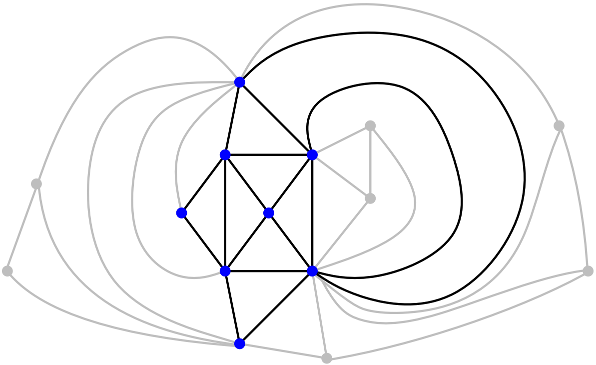

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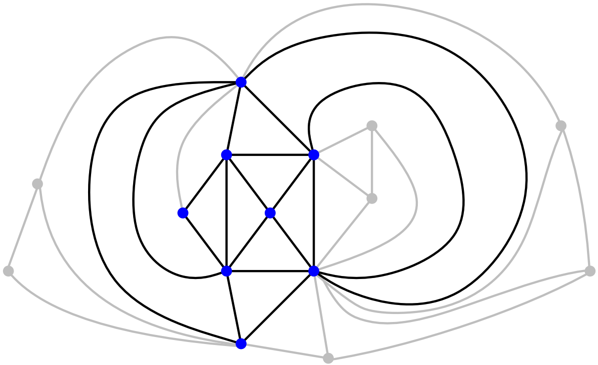

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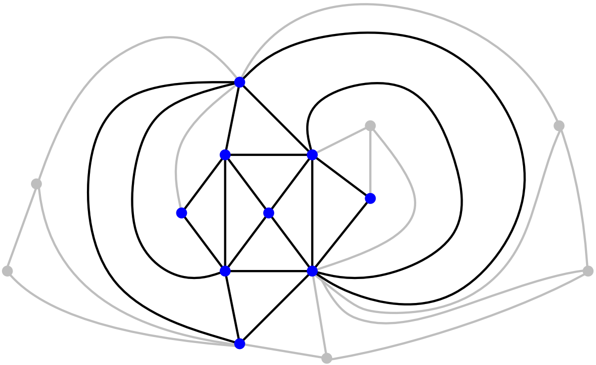

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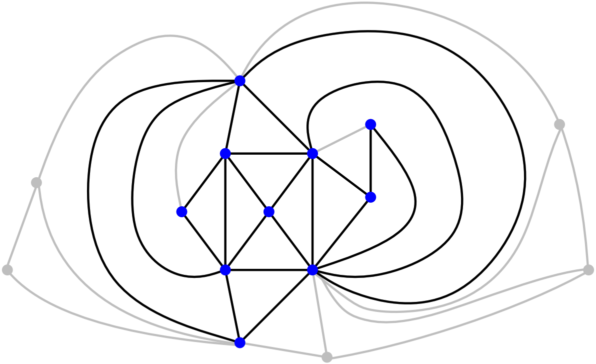

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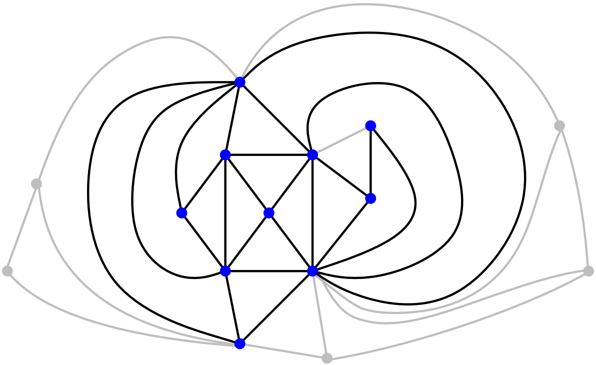

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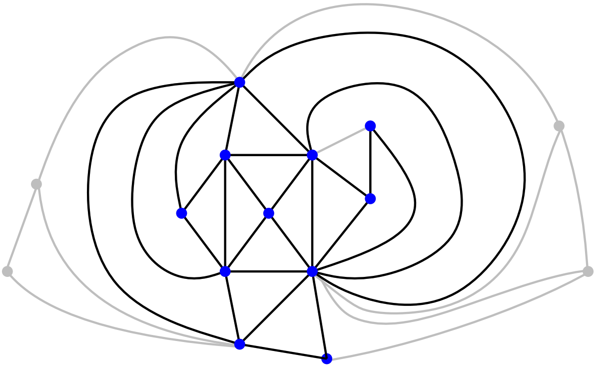

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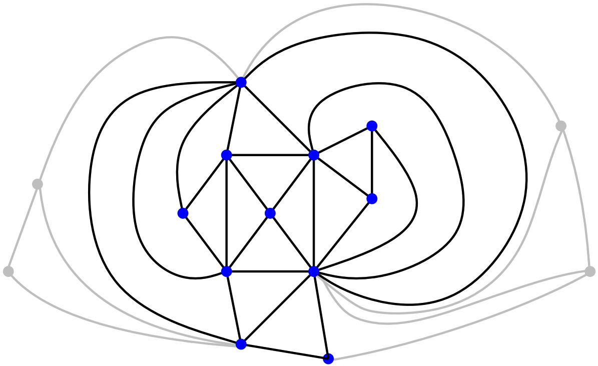

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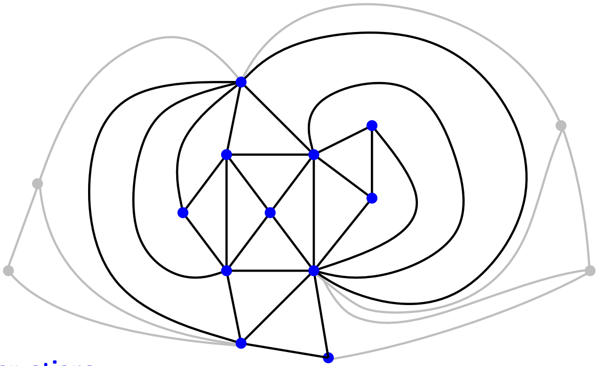

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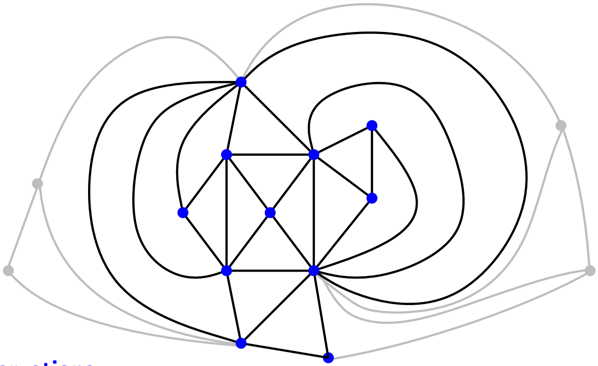

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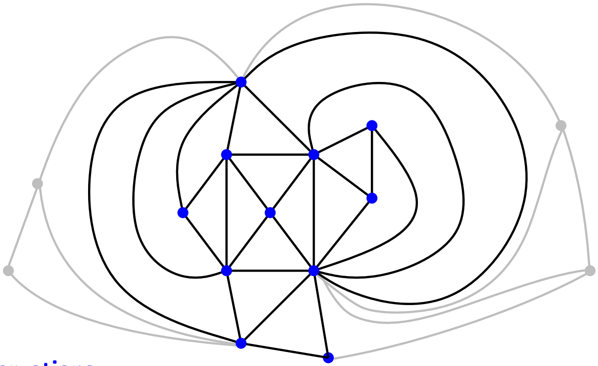

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Belief: Isotropic enough so that at large scales this is close to a ball in the graph metric (now **proved** by Curien and Le Gall)

First passage percolation on random planar maps II

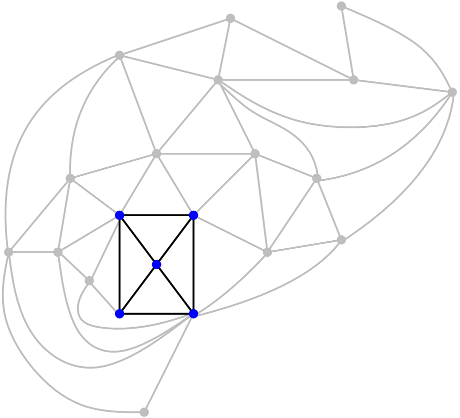
Goal: Make sense of FPP in the continuum on top of a LQG surface

- We do not know how to take a continuum limit of FPP on a random planar map and couple it directly with LQG
- Explain a discrete variant of FPP that involves two operations that we do know how to perform in the continuum:
  - Sample random points according to boundary length
  - Draw (scaling limits of) critical percolation interfaces (SLE_6)

FPP on random planar maps II

Variant:

- Pick two edges on outer boundary of cluster



FPP on random planar maps II

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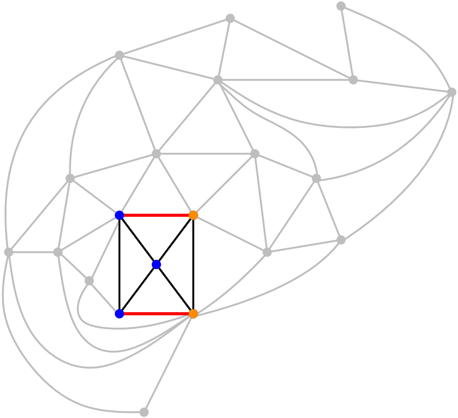

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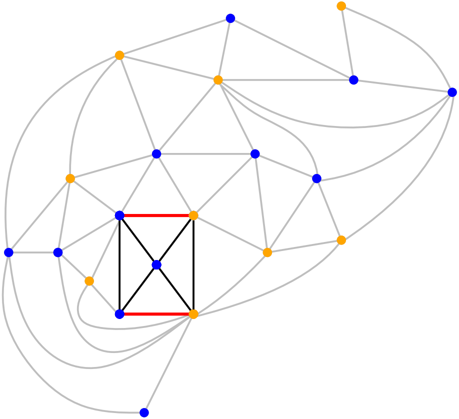

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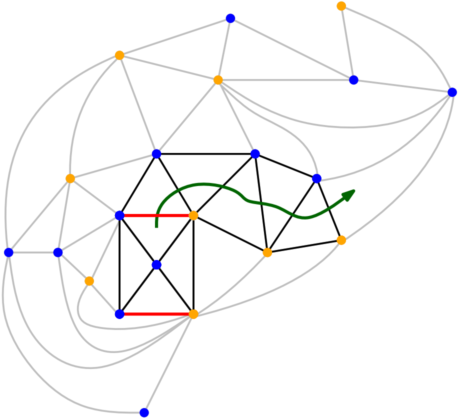

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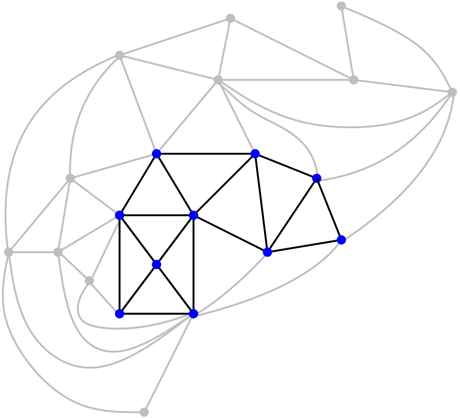

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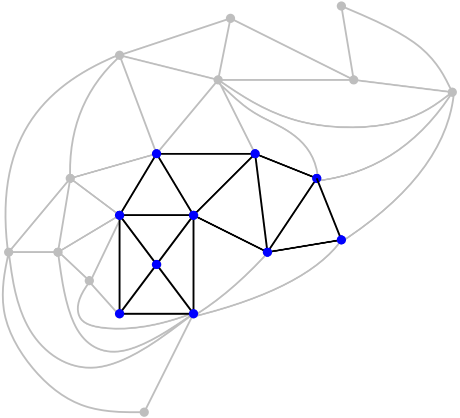

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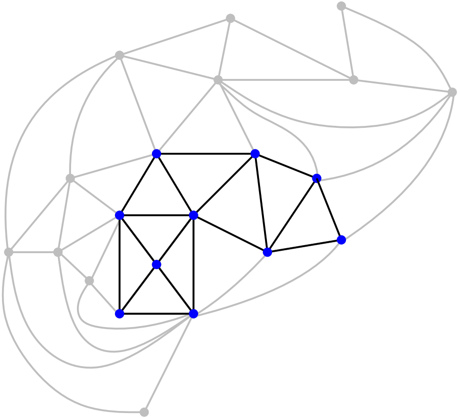

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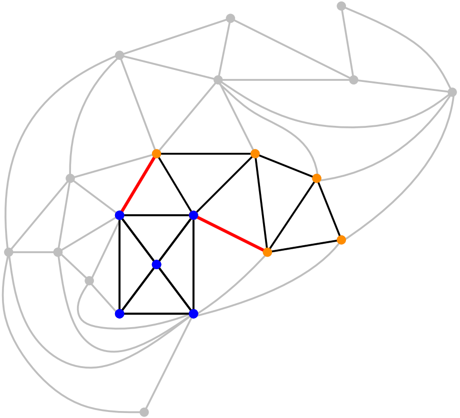

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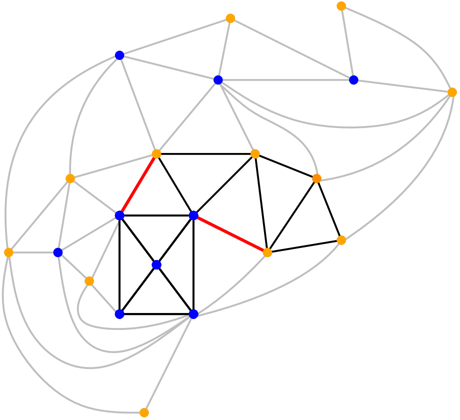

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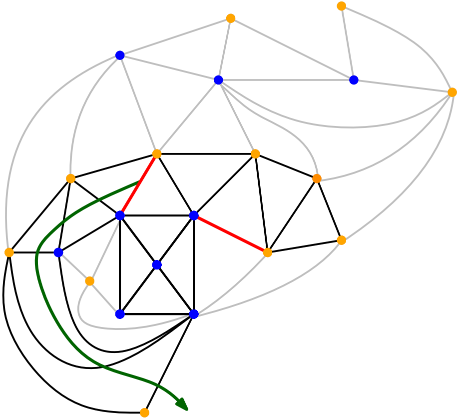

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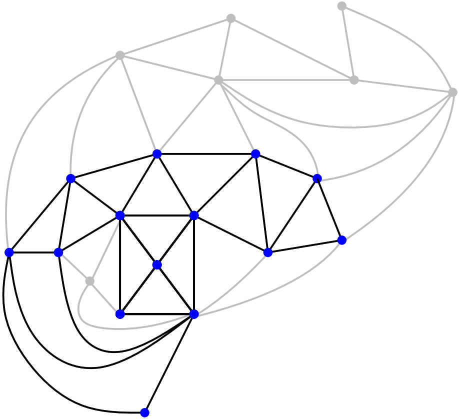

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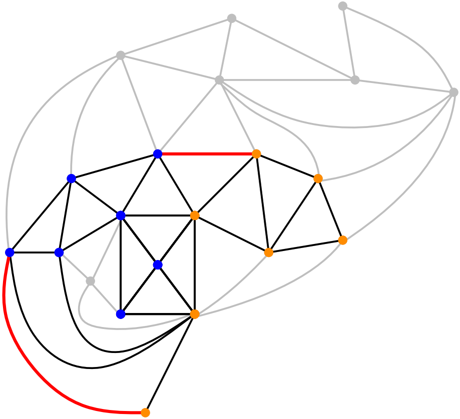

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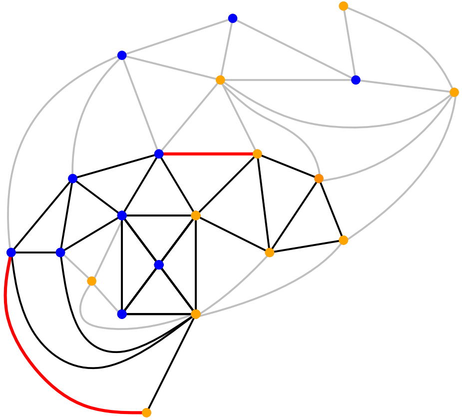

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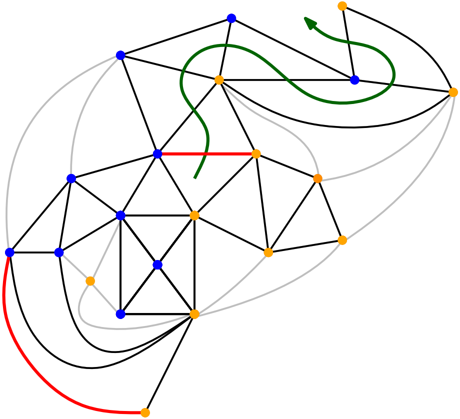

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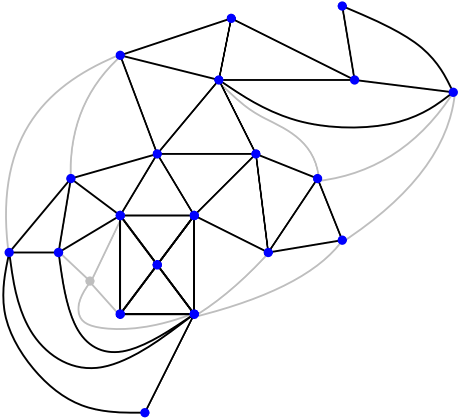

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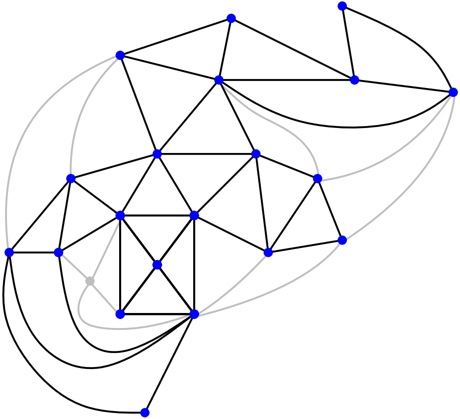

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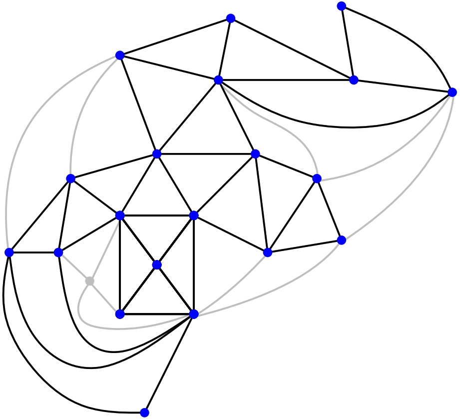
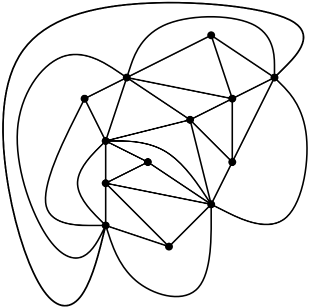


- *This exploration also respects the Markovian structure of the map.*
- Expect that at large scales this growth process looks the same as FPP, hence the same as the graph metric ball

Continuum limit ansatz



- Sample a random planar map

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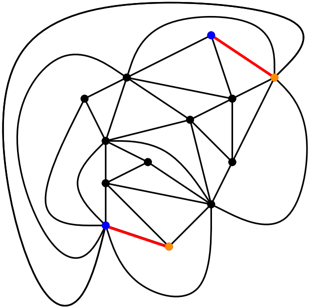

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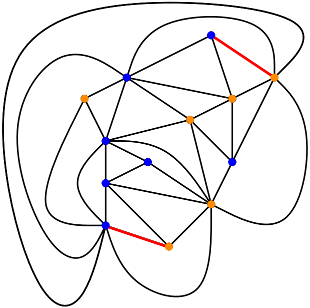

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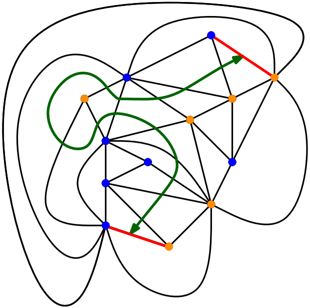

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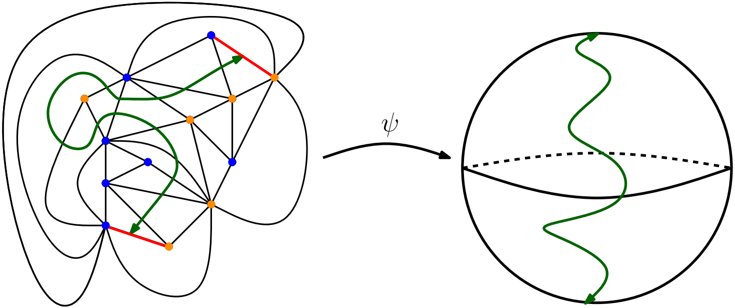

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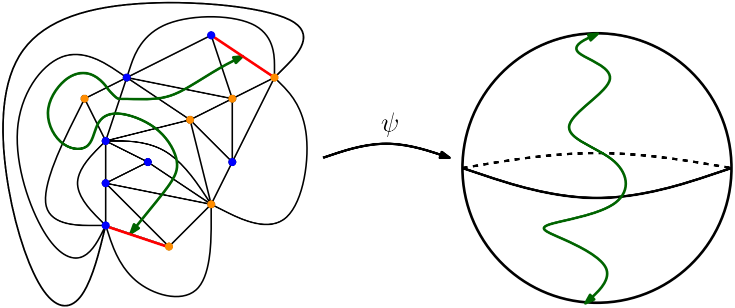

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Ansatz Image of random map converges to a $\sqrt{8/3}$-LQG surface and the image of the interface converges to an independent SLE_6.

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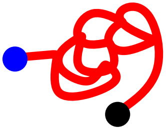

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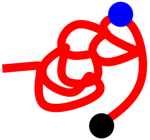

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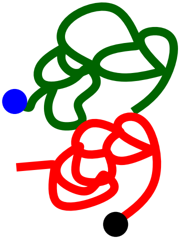

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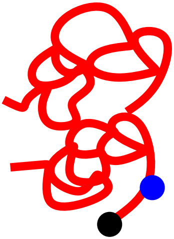

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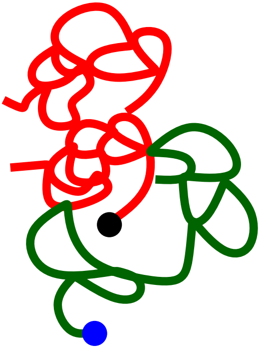

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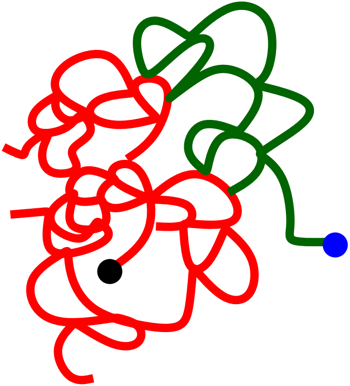

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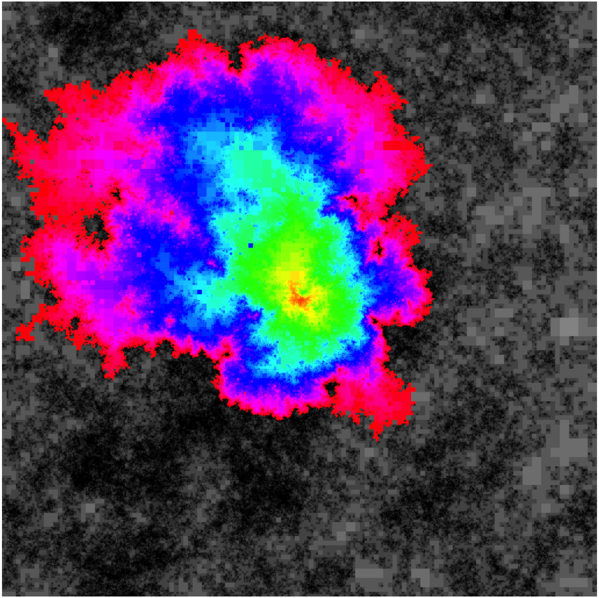
$\mathrm{QLE}(8/3, 0)$ is SLE_6 with **tip re-randomization**.



Discrete approximation of $\mathrm{QLE}(8/3, 0)$. Metric ball on a $\sqrt{8/3}$-LQG

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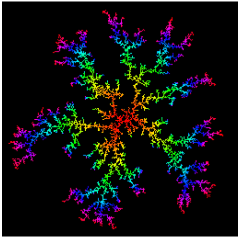
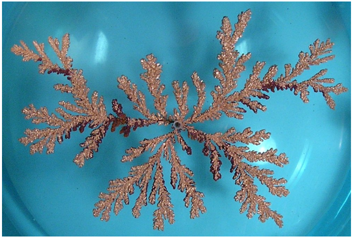
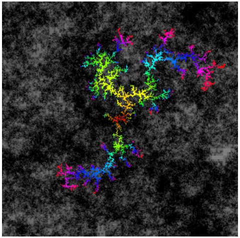
- Still a lot of work to show that resulting metric space structure has the law of TBM and that $\sqrt{8/3}$-LQG and TBM are measurable with respect to each other. But can start to see the Brownian map structure emerge: boundary lengths of metric balls in both spaces evolve in the same way.

Quantum Loewner evolution

$\mathrm{QLE}(8/3, 0)$ is a member of a family of processes which are candidates for the scaling limits of DLA and the dielectric breakdown model on LQG surfaces.



More in **Scott Sheffield's** talk on **Friday**.

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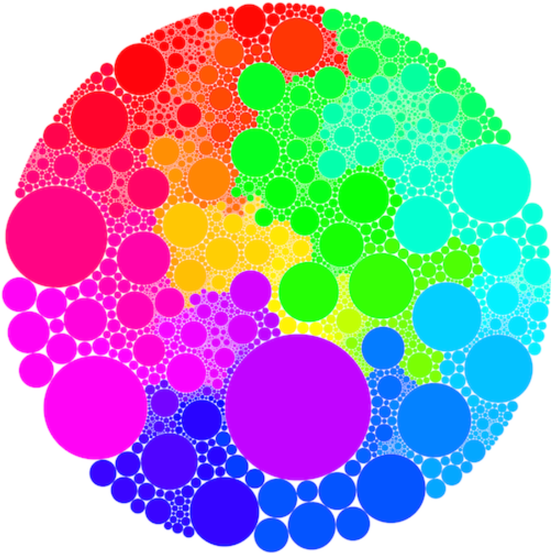

Thanks!