CLE Percolations

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Cambridge

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Outline

Part I: Introduction and motivation

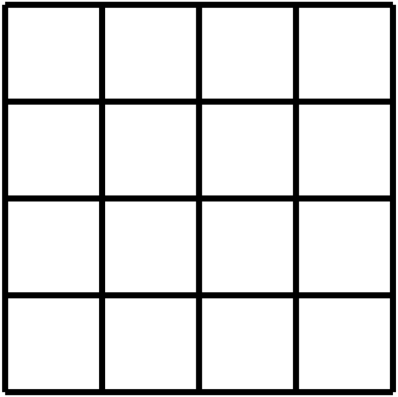
Part II: SLE and CLE

Part III: Conformal percolation and results

Part I: Introduction and motivation

Percolation review

- Graph $G = (V, E)$, $p \in (0, 1)$.



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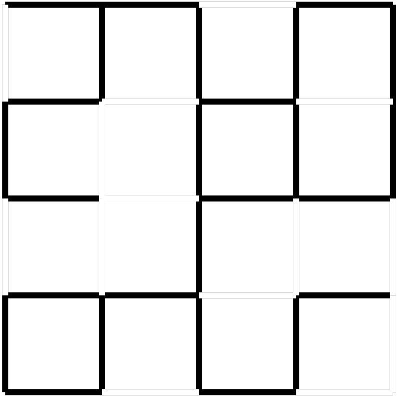

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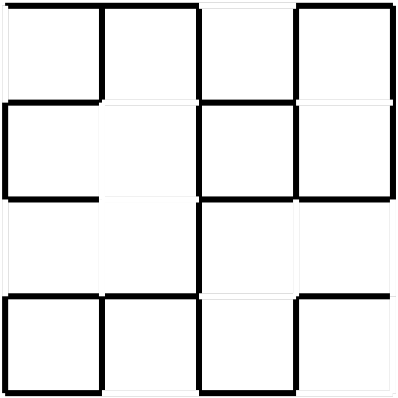

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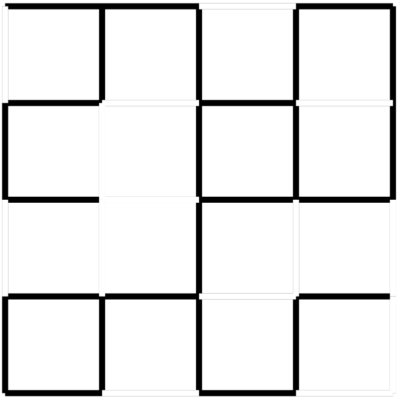

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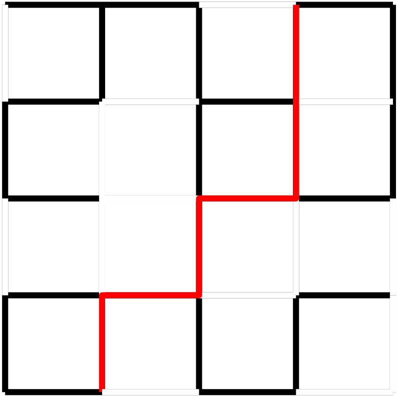

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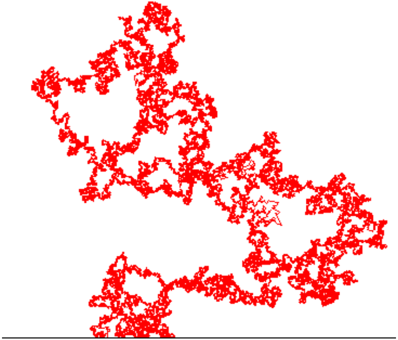

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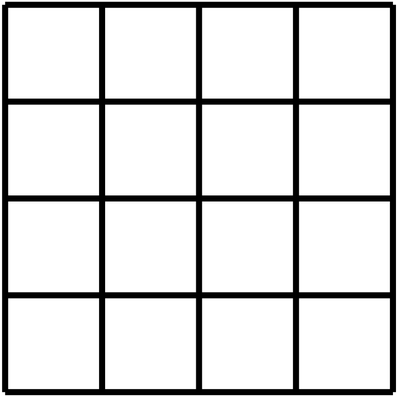
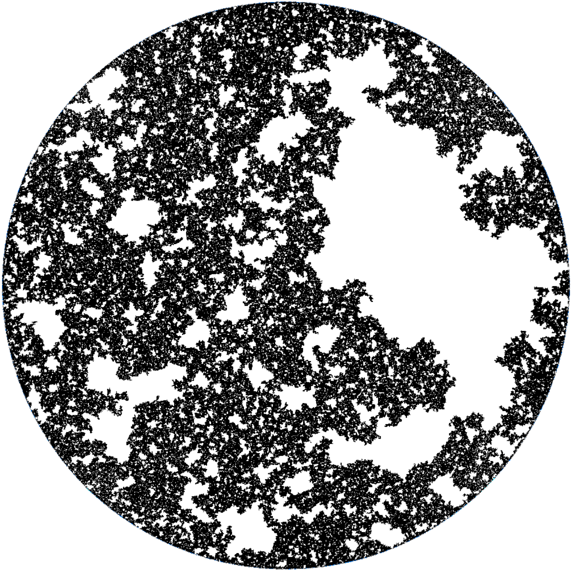
- Leads to better understanding of the underlying graph G

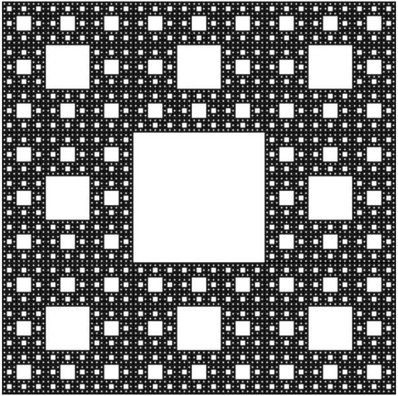




Critical bond percolation on a box in $\mathbf{Z}^2$ with side-length 1000, conformally mapped to $\mathbf{D}$. Shown are the clusters which touch the boundary.

Percolation in fractal carpets

- Can ask the same questions about percolation in a **fractal** carpet or gasket



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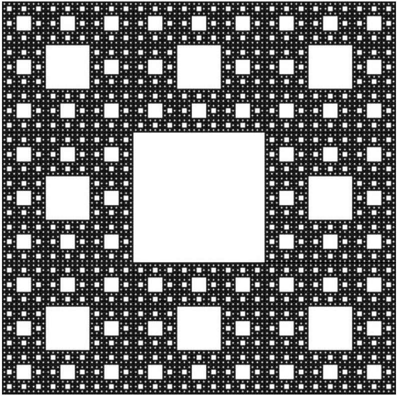

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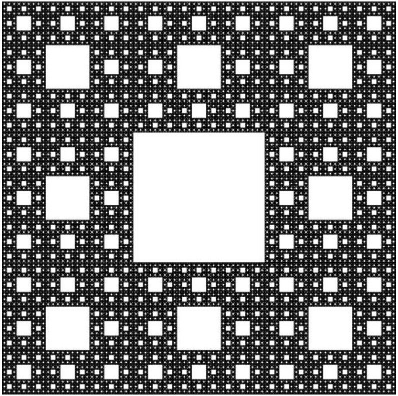

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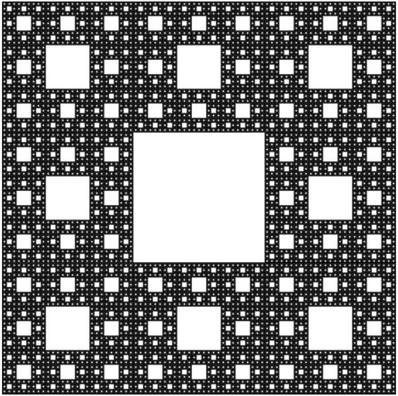

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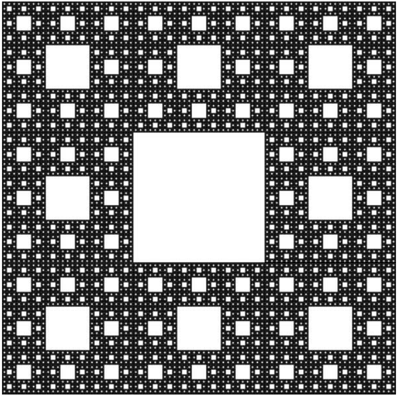

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We will be interested in what happens when the fractal carpet is **random**.

Ising model

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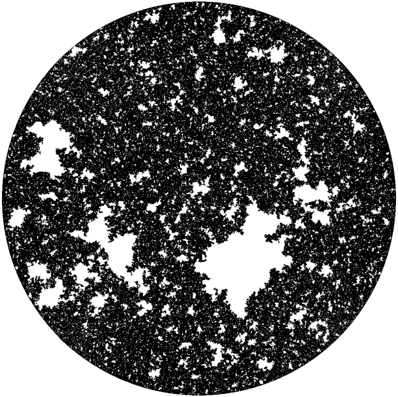

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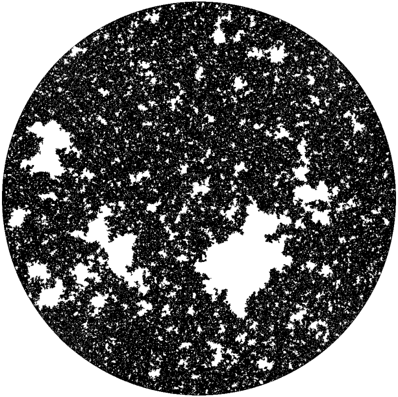
where β is the inverse temperature and $\mathcal{Z}$ is a normalization constant.



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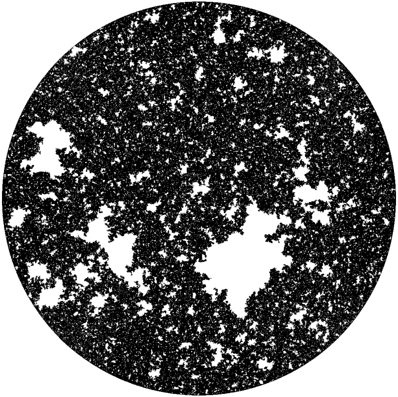
- Model for a magnet.



Ising model with $+$ boundary conditions.

We will be interested in percolation in random fractal carpets which arise from models like the Ising model.

FK random cluster representation of the Ising model

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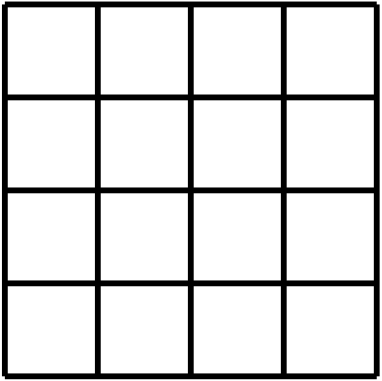

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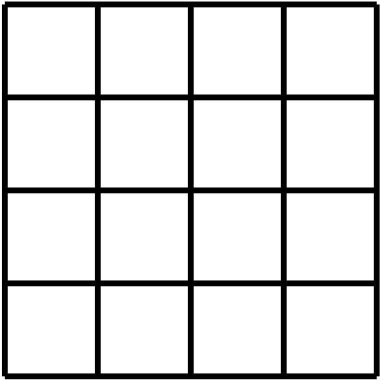

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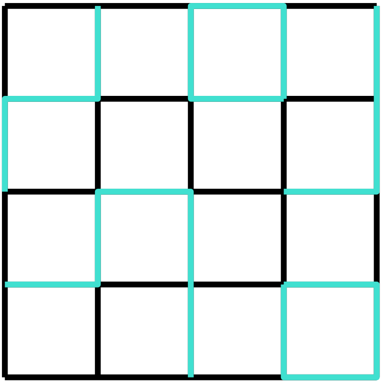

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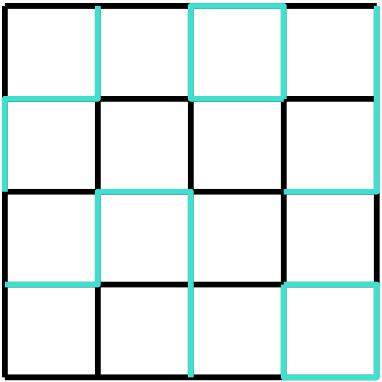

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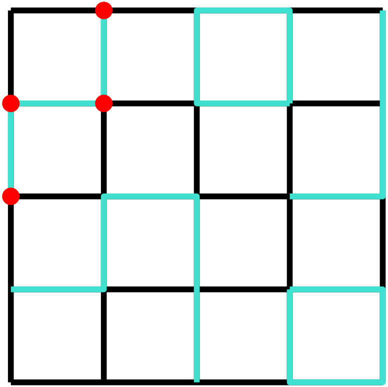

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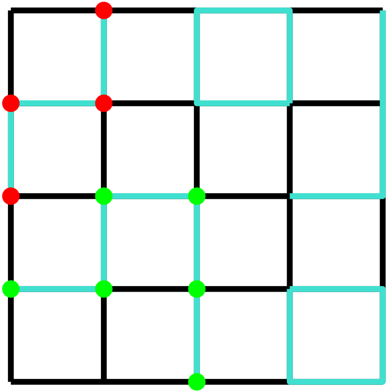

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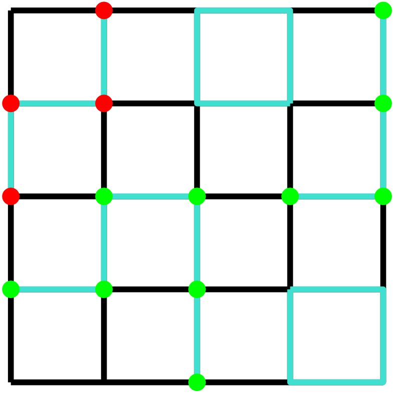

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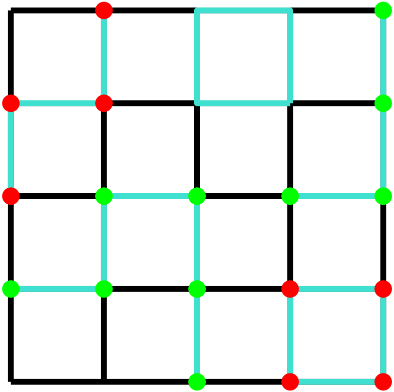

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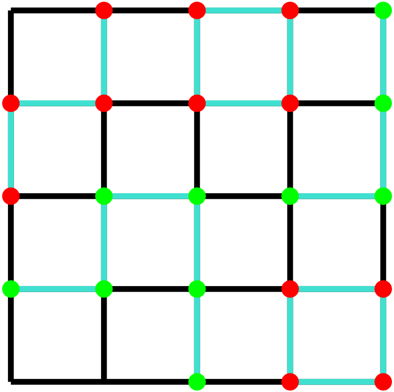

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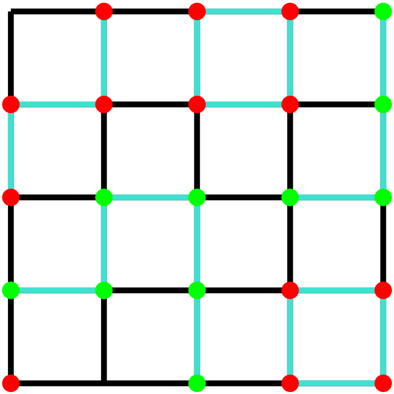

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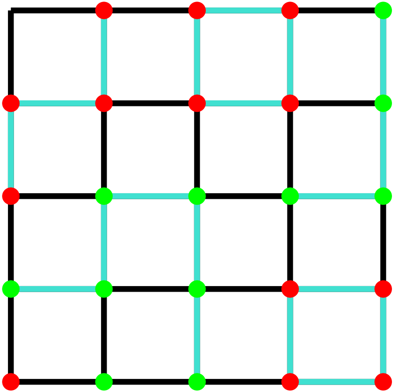

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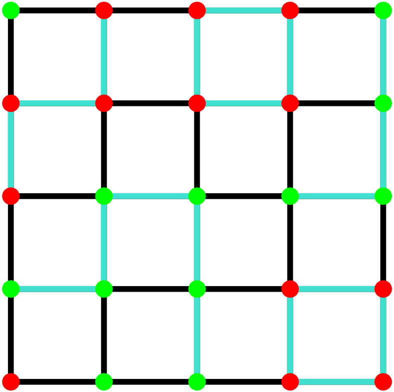

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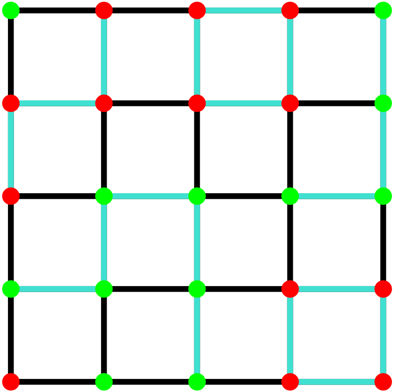
- Given Ising instance, can sample edges by opening edges between same spin sites independently with probability p.



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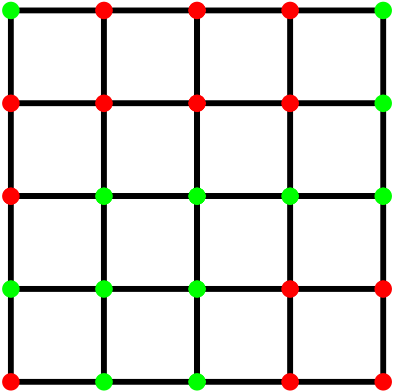

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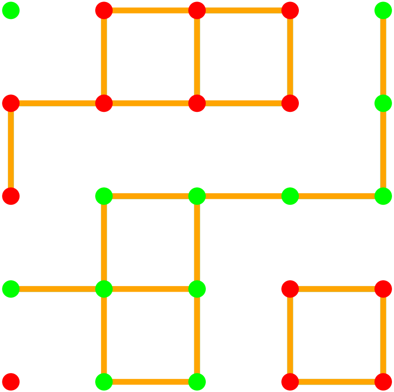

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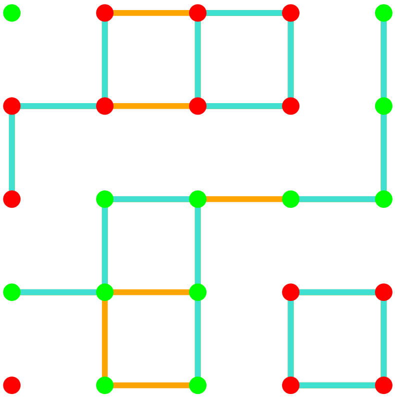

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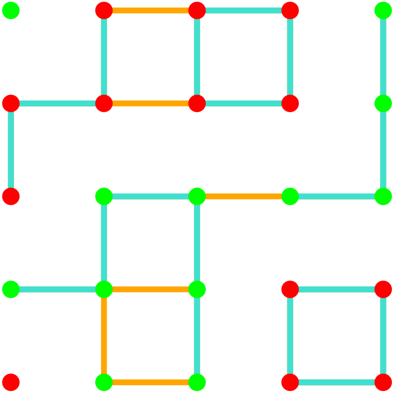

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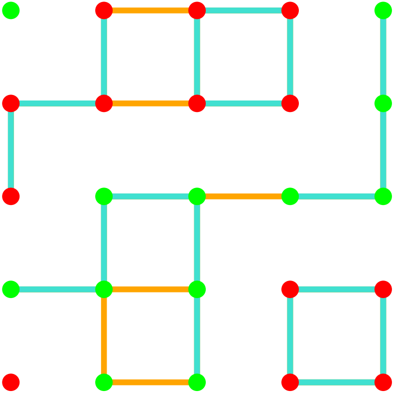
- Generalizes to q-state Potts models.



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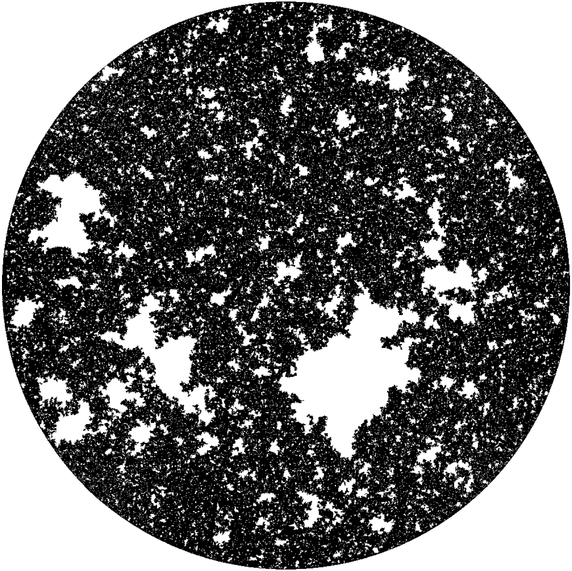
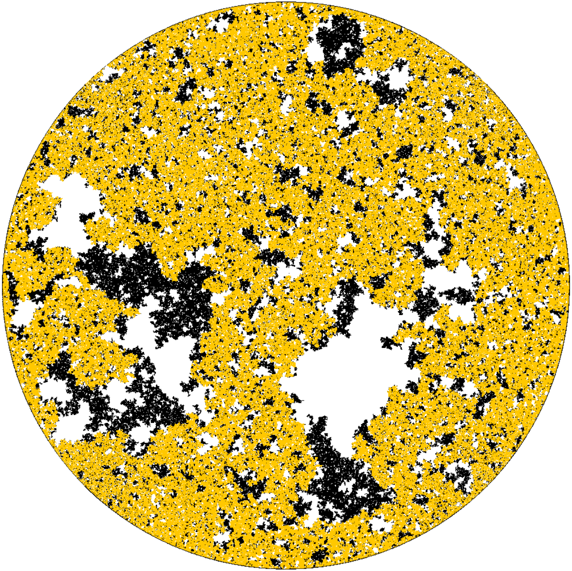
- The full collection of interfaces between $\pm$ sites in the Ising model converge to CLE_3 (Benoist-Hongler).
- There should be a coupling of CLE_3 and $\mathrm{CLE}_{16/3}$ which satisfy the same properties:
  - $\mathrm{CLE}_{16/3} =$ percolation in the CLE_3 carpet
  - $\mathrm{CLE}_3 =$ interfaces between i.i.d. $\pm$-labeled $\mathrm{CLE}_{16/3}$ clusters



Ising model on box in $\mathbf{Z}^2$ of side-length 1000, all $+$ boundary conditions, conformally mapped to $\mathbf{D}$. Boundary touching $+$ cluster in black.



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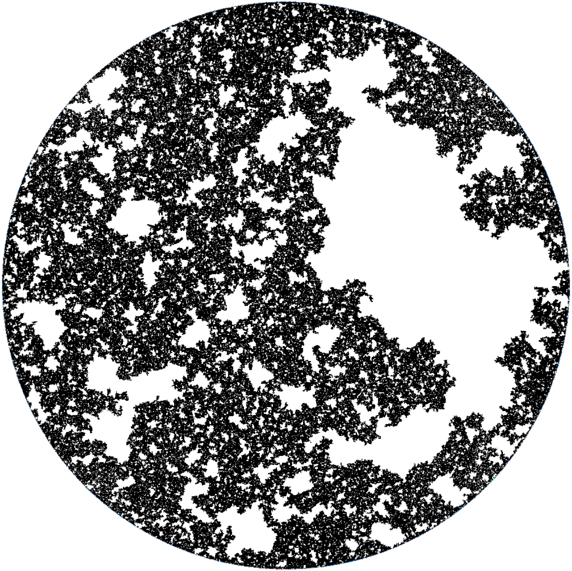
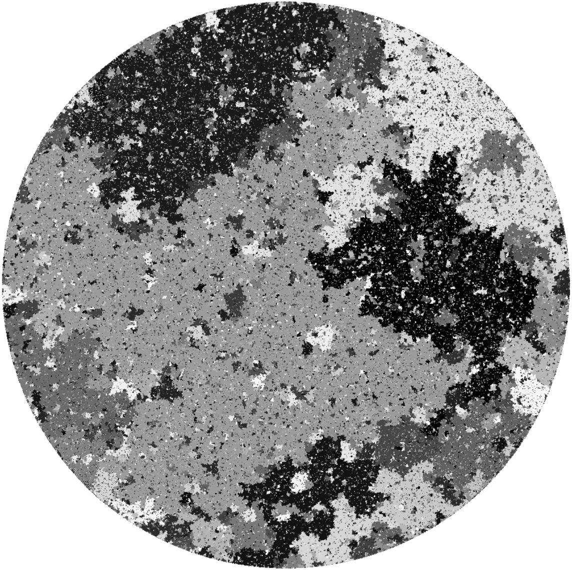


Critical bond percolation on box in $\mathbf{Z}^2$ of side-length 1000. Shown in black are the boundary touching clusters.



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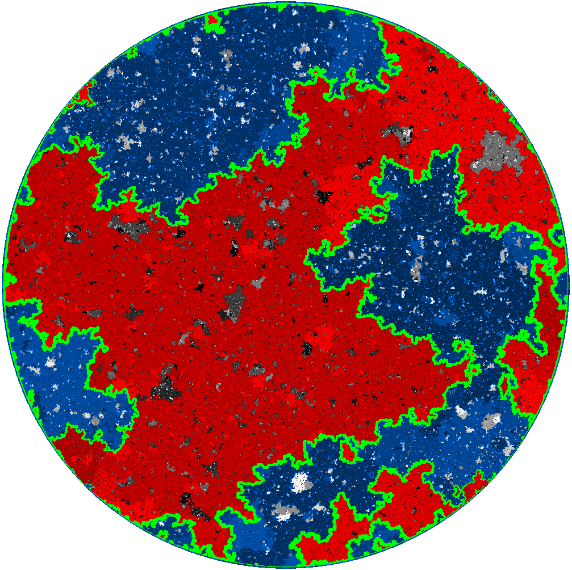
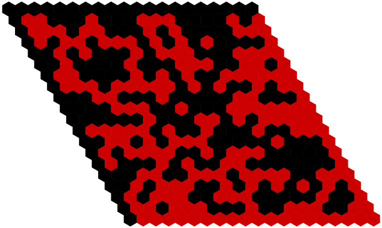


Critical bond percolation on box in $\mathbf{Z}^2$ of side-length 1000. Boundary touching clusters with label $\leq 1/2$ (resp. $> 1/2$) in red (resp. blue). Blue/red interface in green.

Part II: SLE and CLE

Schramm-Loewner evolution (SLE)

- Random fractal curve in a planar domain



Critical percolation, hexagonal lattice

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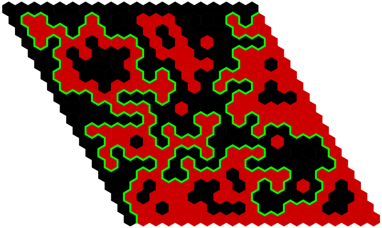

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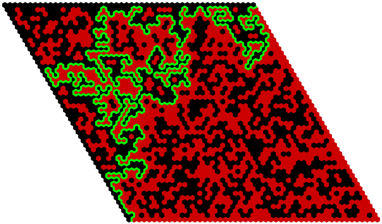

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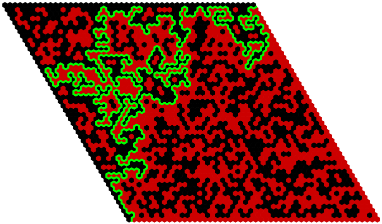

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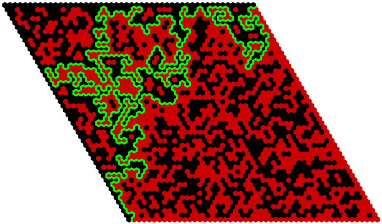

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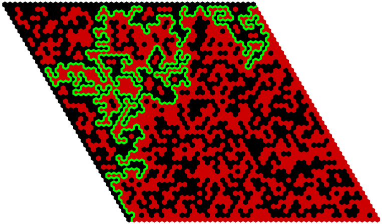

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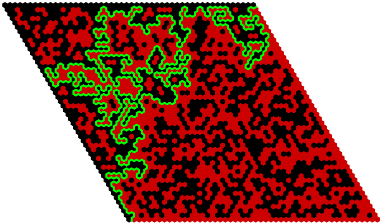

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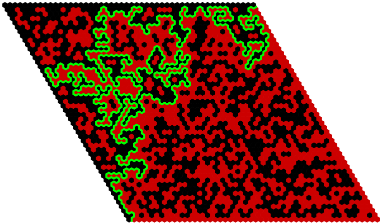
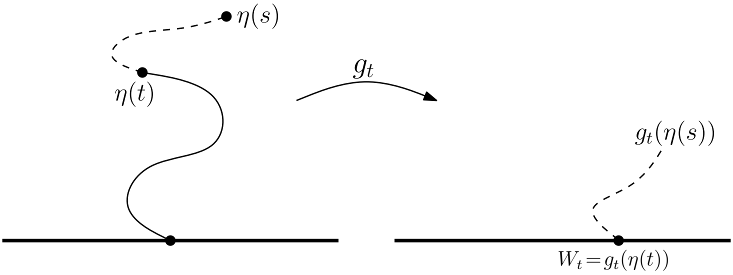


Critical percolation, hexagonal lattice

SLE_κ



Loewner's equation: if η is a non self-crossing path in $\mathbf{H}$ with $\eta(0) \in \mathbf{R}$ and g_t is the Riemann map from the unbounded component of $\mathbf{H} \setminus \eta([0,t])$ to $\mathbf{H}$ normalized by $g_t(z) = z + o(1)$ as $z \to \infty$, then

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t} \text{ where } g_0(z) = z \text{ and } W_t = g_t(\eta(t)). \qquad (\bigstar)$$

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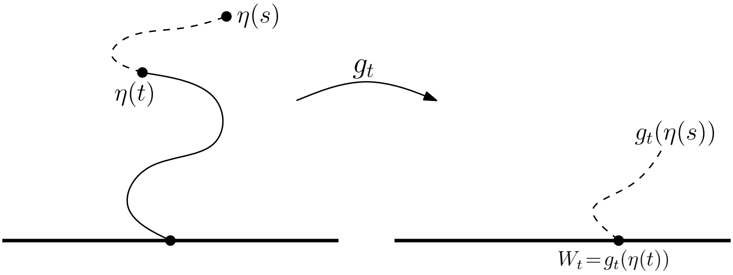

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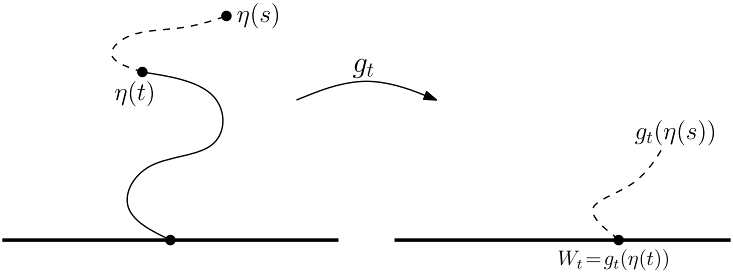

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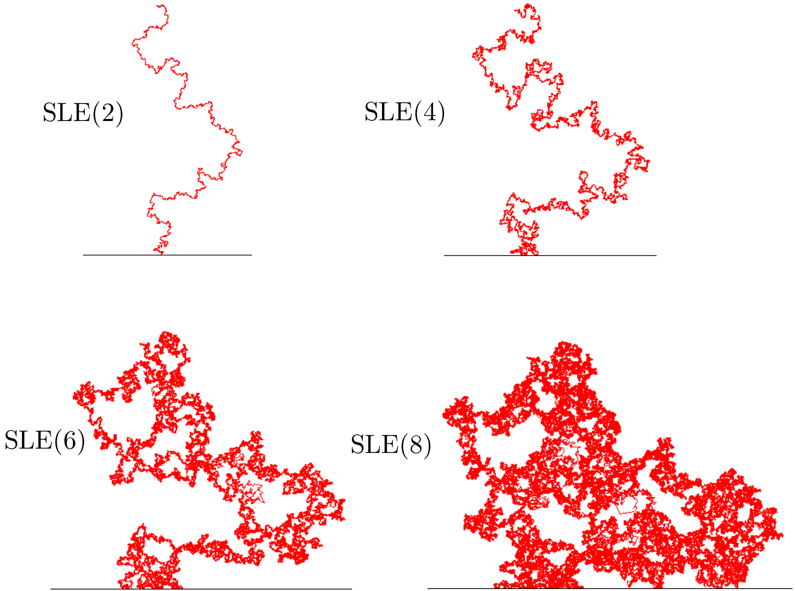
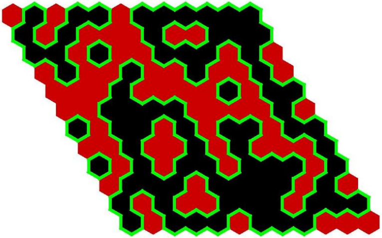


Simulations due to Tom Kennedy.

Conformal loop ensembles (CLE)

- Random collection of loops in a planar domain. Loop version of SLE.



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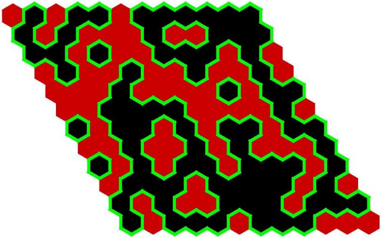

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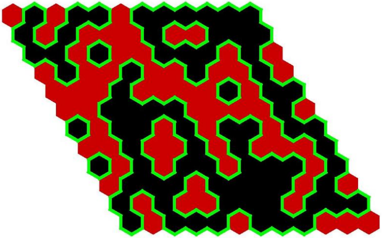

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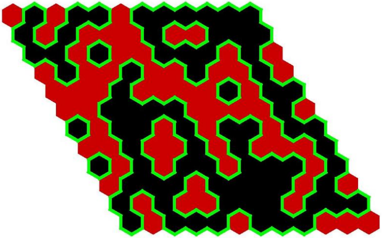

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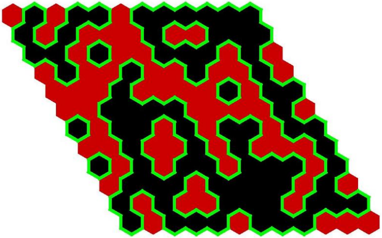

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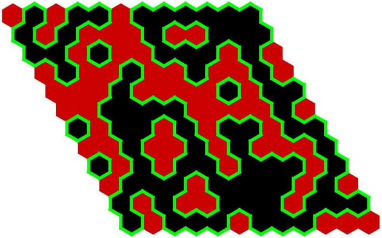

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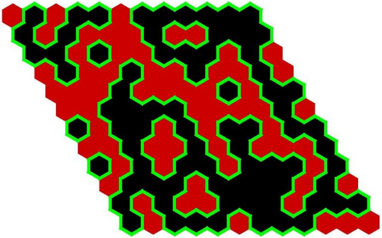

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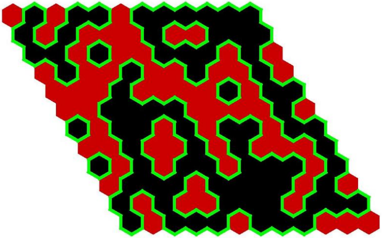

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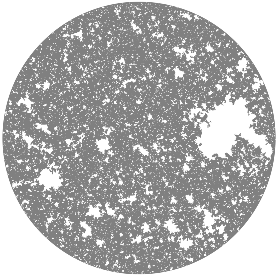
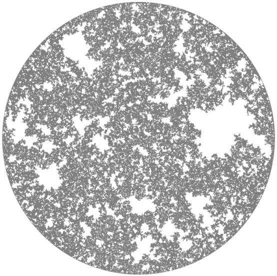
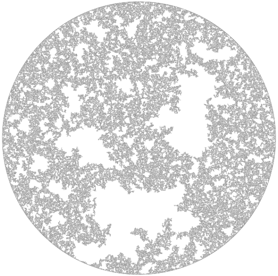


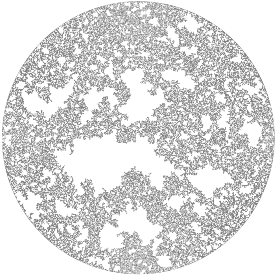
CLE_3



CLE_4



$\mathrm{CLE}_{16/3}$



CLE_6

Simulations due to David B. Wilson.

Part III: Conformal percolation and results

Conformal percolation

- **Goals:**

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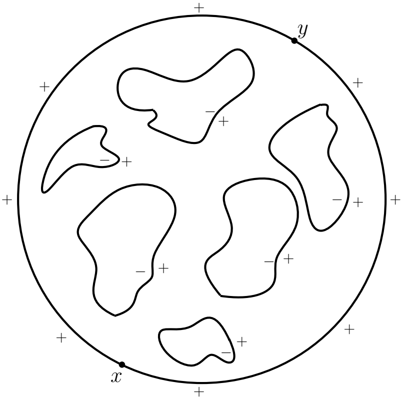
- Then we will have made sense of the random cluster representation of the Ising model in the continuum and, more generally, for all of the Potts models.

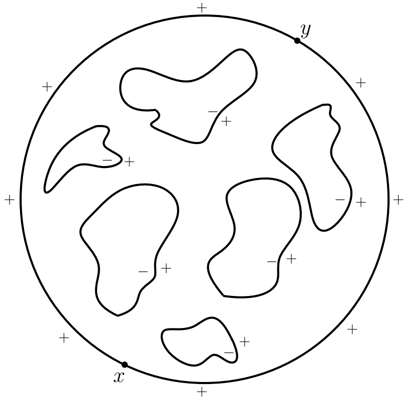
$\mathrm{CLE}_{16/3}$ as percolation in the CLE_3 carpet



CLE_3: scaling limit of outermost $\pm$ Ising interfaces, $+$ boundary conditions

$\mathrm{CLE}_{16/3}$ as percolation in the CLE_3 carpet

- Percolation exploration in CLE_3 carpet from x to y



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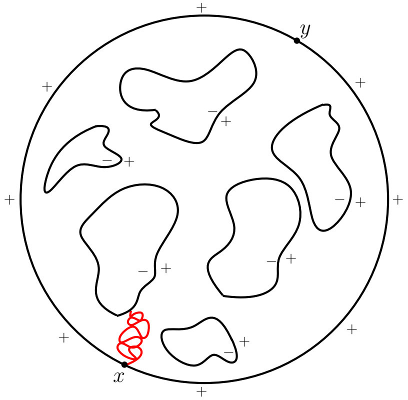

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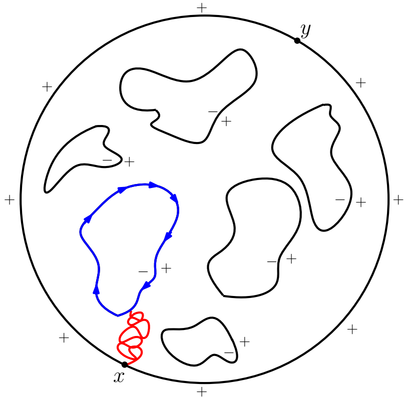

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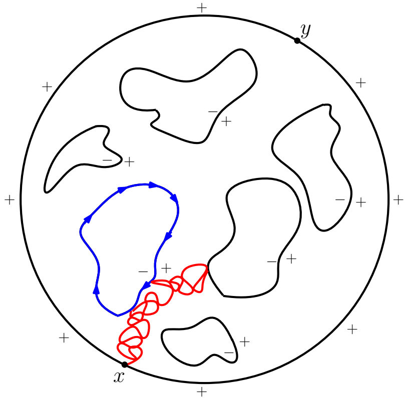

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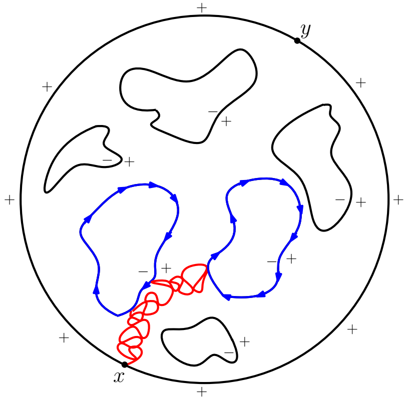

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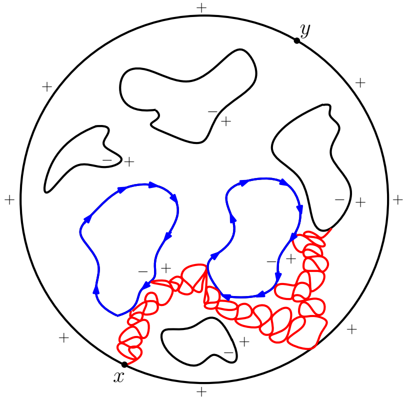

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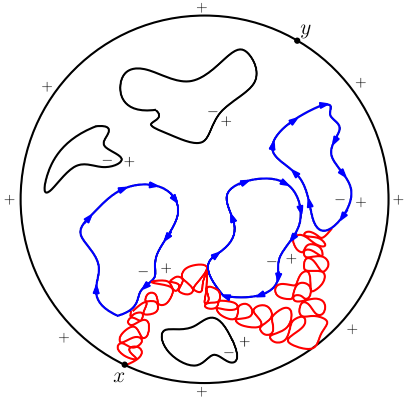

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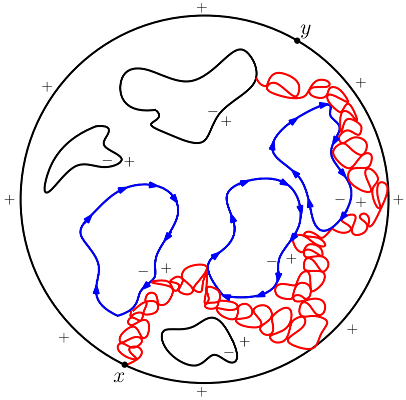

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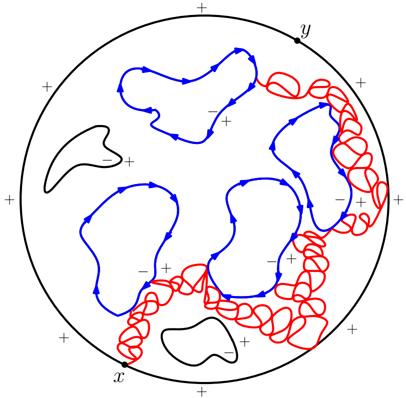

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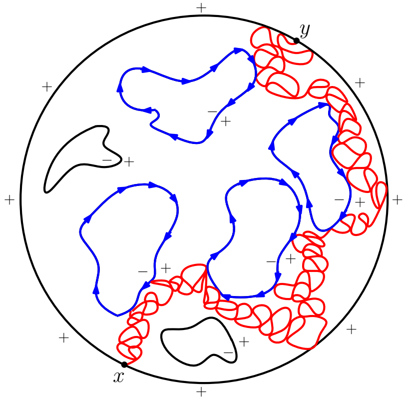

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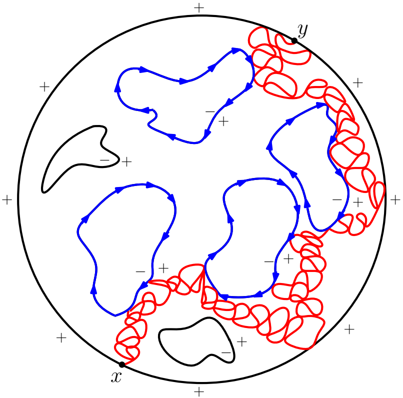

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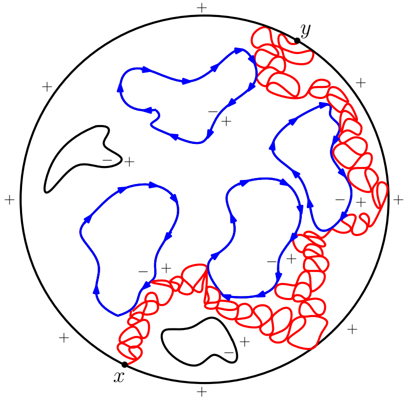

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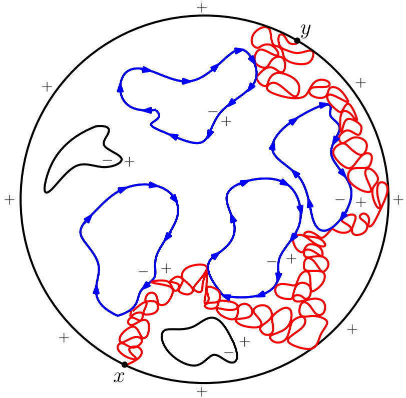

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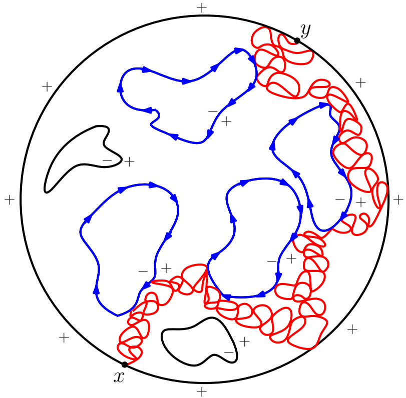

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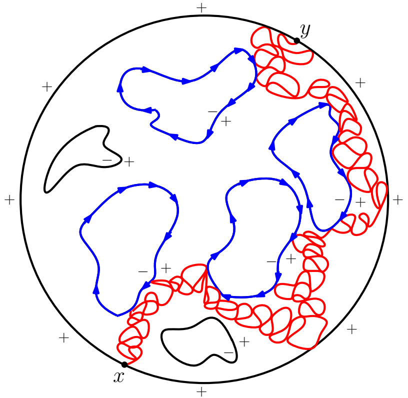

CLE_3: scaling limit of outermost $\pm$ Ising interfaces, $+$ boundary conditions

$\mathrm{CLE}_{16/3}$ as percolation in the CLE_3 carpet

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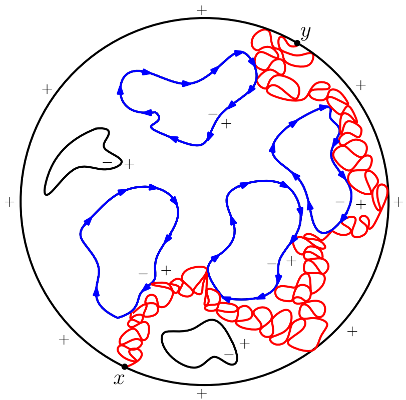

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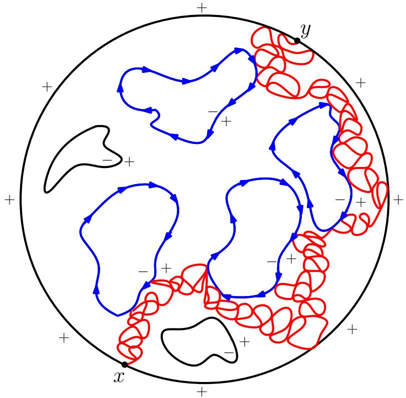

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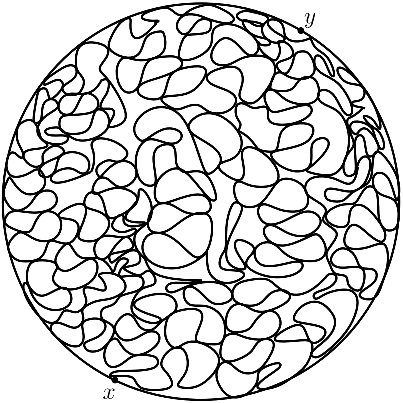

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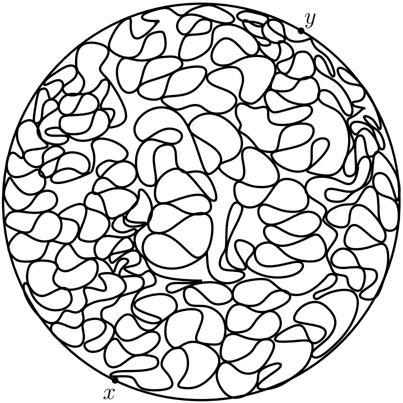

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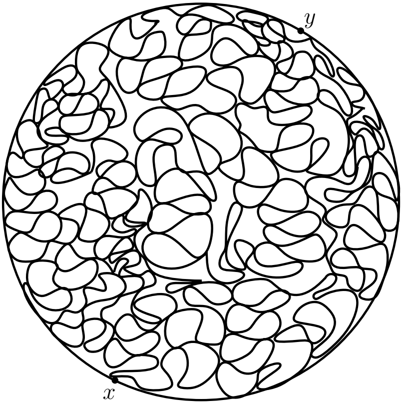

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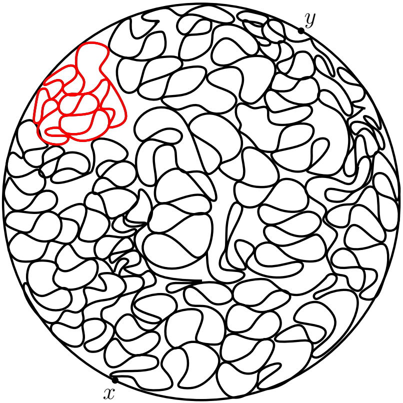

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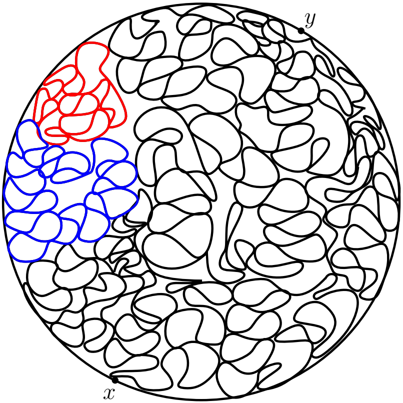

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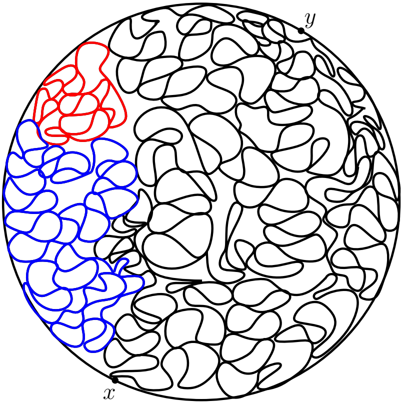

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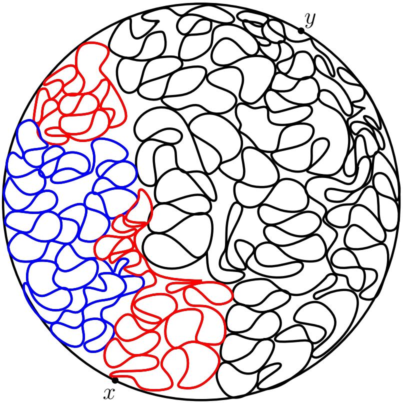

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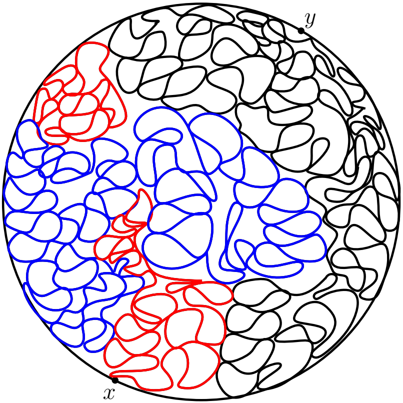

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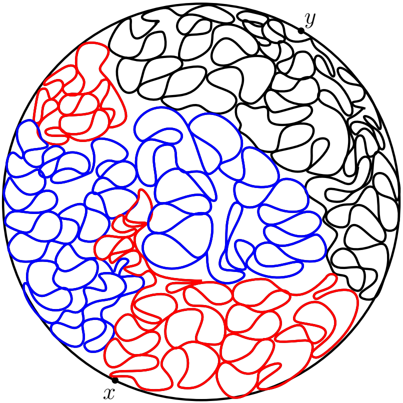

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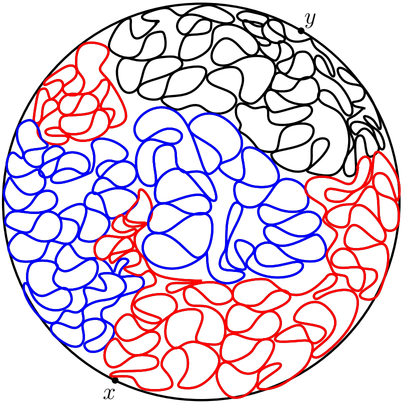

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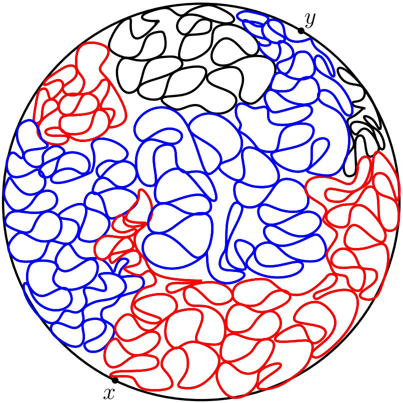

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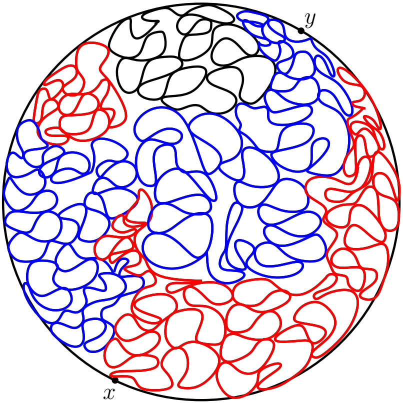

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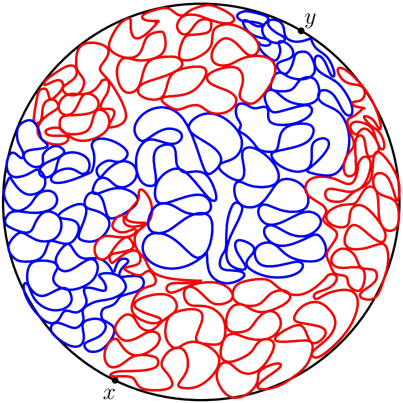

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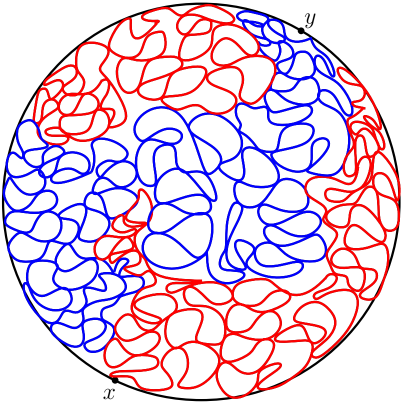

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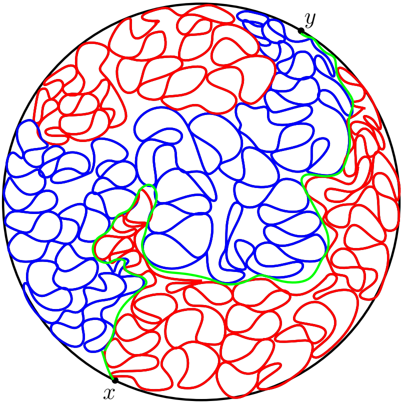

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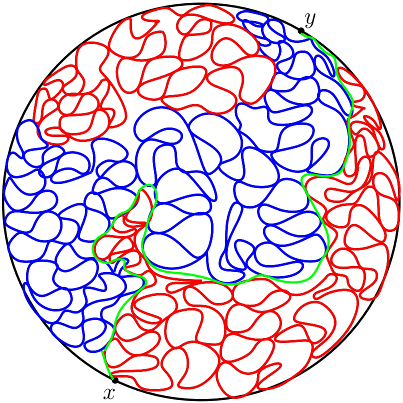

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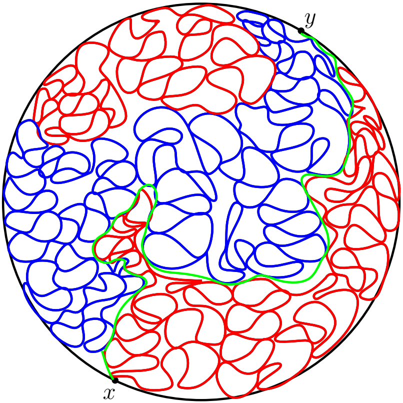

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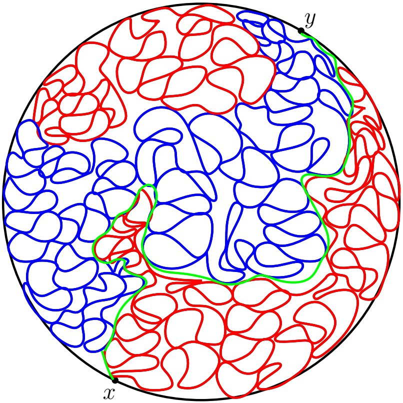

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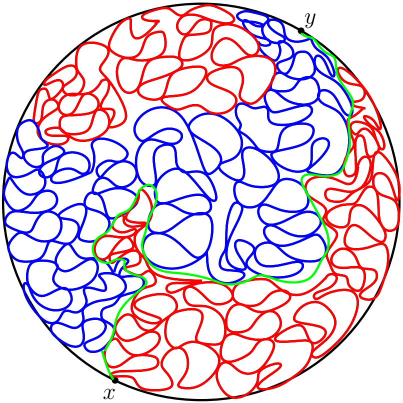

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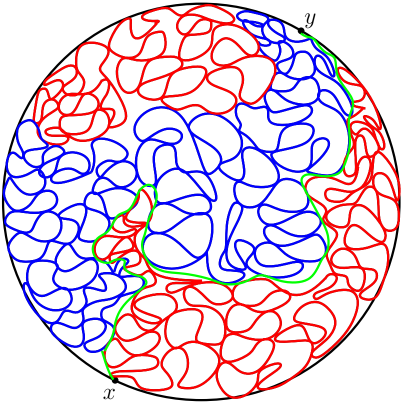

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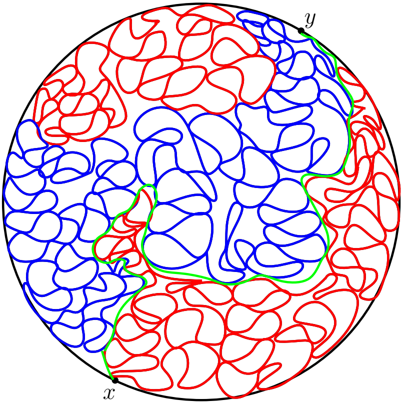

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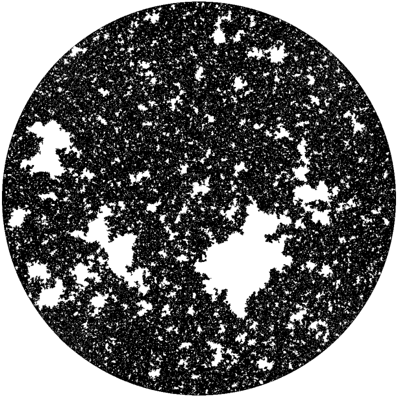

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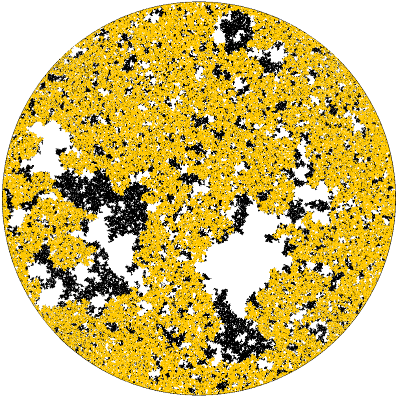

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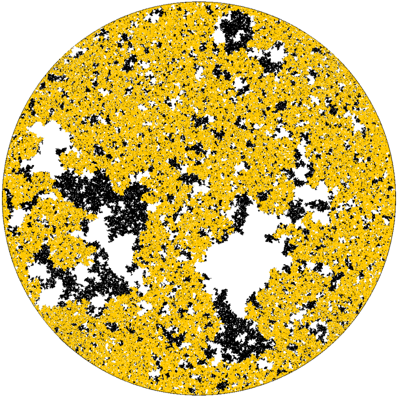

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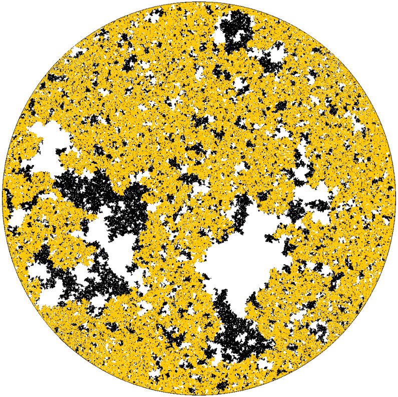

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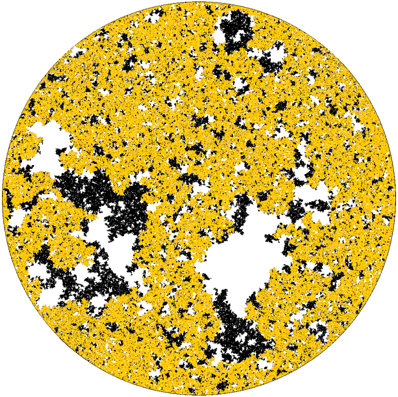

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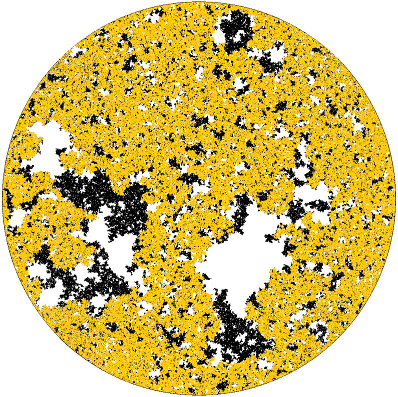

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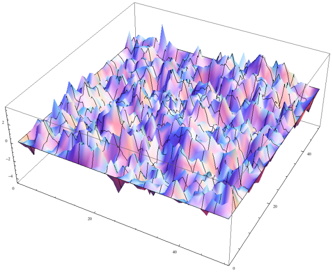
Proof technique



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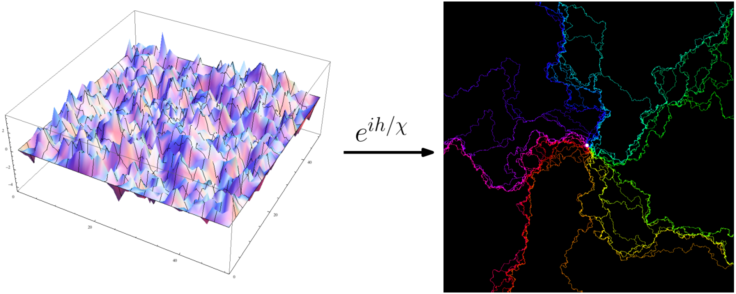

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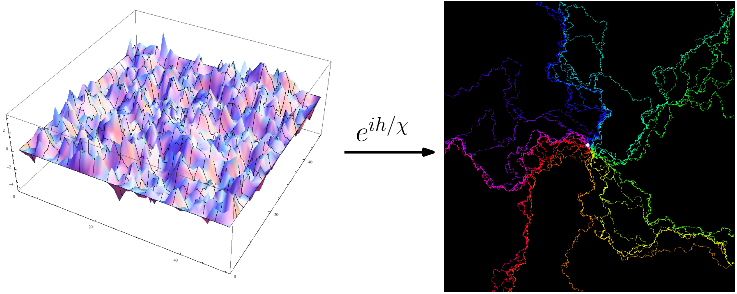

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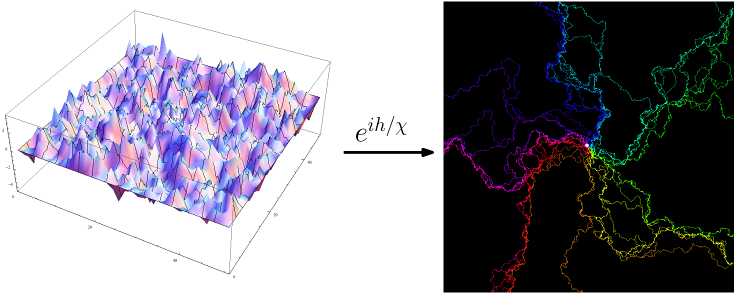

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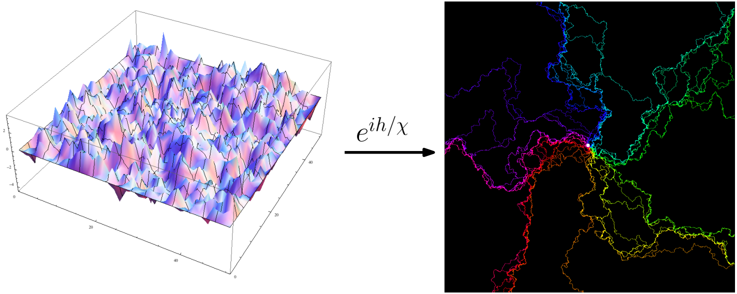

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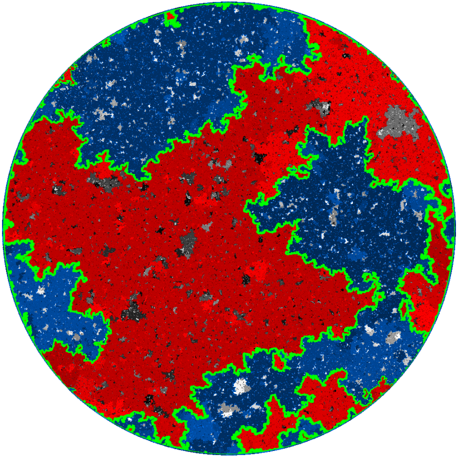
- The coupling of CLE_3 and $\mathrm{CLE}_{16/3}$ turns out to sit exactly in this framework
- The same is true more generally for CLE_κ, $\kappa \in (8/3, 4]$ and $\mathrm{CLE}_{16/\kappa}$, $16/\kappa \in [4, 6)$



Thanks!