Lent Term 2008 J. M. E. Hyland

Set Theory and Logic: Example Sheet 2

1. (i) Show that a totally ordered set $(A, <)$ is well-ordered if and only if there are no infinite descending sequence $a_0 > a_1 > a_2 > \cdots$. Have you used AC in your proof?
   (ii) Write out a proof that a countable union of countable sets is countable. Where is the use of AC in your proof?

2. A collection $\mathcal{X} \subseteq P(X)$ of subsets of a set X, which is such that $A \in \mathcal{X}$ if and only for all finite $a \subseteq A$, $a \in \mathcal{X}$ is said to have *finite character*.
   The Teichmuller-Tukey Lemma is the statement: if $\mathcal{X}$ has finite character then it has a maximal element.
   Show that AC implies the Teichmuller-Tukey Lemma. Does the Teichmuller-Tukey Lemma in its turn imply AC?

3. Consider the statement: for any pair of cardinals $\mathbf{m}$ and $\mathbf{n}$, either $\mathbf{m} \leq \mathbf{n}$ or $\mathbf{n} \leq \mathbf{m}$. Show that it is equivalent to AC.

4. (This question is a curiosity, not essential to the course.)
   Let AC_n be the principle that any family of n-element sets has a choice function.
   (i) Show that AC_{rs} implies AC_r for all $r, s \geq 1$.
   (ii) Show that AC_2 implies AC_4.

5. A complete poset $(X, \leq)$ is one in which the supremum $\sup A$ exists for all $A \subseteq X$. Show that in a complete poset the infimum $\inf A$ also exists for all $A \subseteq X$.
   Show that the fixed point constructed in the proof of the Knaster-Tarski Theorem is the greatest fixed point. Modify the proof to produce the least fixed point instead.

6. Which of the following posets (ordered by inclusion) are complete?
   (i) The set of all subsets of $\mathbb{N}$ that are finite or have finite complement.
   (ii) The set of all independent subsets of a vector space V.
   (iii) The set of all subspaces of a vector space V.
   (iv) The set of all equivalence relations $R \subseteq X \times X$ on a set X.

7. Let X be a complete poset, and let $f : X \to X$ be order-reversing (meaning that $x \leq y$ implies $f(x) \geq f(y)$). Give an example to show that f need not have a fixed point. Show, however, that there must exist either a fixed point of f or two distinct points x and y with $f(x) = y$ and $f(y) = x$.

8. (i) What is the cardinality of the set of open subsets in $\mathbb{R}$?
   (ii) What is the cardinality of the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$?

9. Show that any two bases of a vector space have the same cardinality. Did you use AC?

10. Define the sum $\sum_{i \in I} L_i$ and product $\prod_{i \in I} L_i$ of an indexed family $(L_i \mid i \in I)$ of sets.
    Suppose that $(L_i \mid i \in I)$ and $(M_i \mid i \in I)$ are such that there are no surjections $L_i \to M_i$ for any $i \in I$. Show that there is no surjection $\sum_{i \in I} L_i \to \prod_{i \in I} M_i$.
    Deduce that there is no surjection from $\aleph_\omega$ to ${\aleph_\omega}^{\aleph_0}$. Can we have the equality $2^{\aleph_0} = \aleph_\omega$?

11. (Aspects of cardinal arithmetic without choice, so not central to the course.)
(i) Prove that, even without AC, a countable union of countable sets certainly cannot have cardinality $\aleph_2$. (This should encourage quiet reflection on question 1 (ii).)
(ii) Show that the cardinality of the set of well-orderings of the set $\mathbb{N}$ is $2^{\aleph_0}$.
Deduce that, even without AC, $\aleph_1 \leq^* 2^{\aleph_0}$.

12. (i) Show that if $\mathbf{m} + \mathbf{n} = \mathbf{m}.\mathbf{n}$, then either $\mathbf{n} \leq \mathbf{m}$ or $\mathbf{m} \leq^* \mathbf{n}$.
(ii) Take κ a well-ordered cardinal (or initial ordinal). Show that if $\kappa + \mathbf{n} = \kappa.\mathbf{n}$, then either $\mathbf{n} \leq \kappa$ or $\kappa \leq \mathbf{n}$.
(iii) Deduce that if $\mathbf{m} + \mathbf{n} = \mathbf{m}.\mathbf{n}$ for all infinite cardinals $\mathbf{m}$ and $\mathbf{n}$, then AC holds.

13. Is there an ordinal α such that $\omega_\alpha = \alpha$?

14. Suppose that κ is an aleph (that is an infinite well-ordered cardinal). Show that if $\kappa \leq \mathbf{m}.\mathbf{n}$ then either $\kappa \leq \mathbf{m}$ or $\kappa \leq \mathbf{n}$.

15. How many different partial orders (up to isomorphism) are there on a set of 4 elements? How many of these are complete?

16. An open set in a topological space is said to be *regular* when it is the interior of its closure.
Give an example of a regular and a non-regular open set in $\mathbb{R}$.
Is the union of two regular opens necessarily regular? Is the intersection of two regular opens necessarily regular?
Consider the collection $\mathcal{R}(X)$ of regular open sets in a space X order by inclusion.
Show that $\mathcal{R}(X)$ is a boolean algebra.
Is $\mathcal{R}(X)$ a complete poset?
Is $\mathcal{R}(\mathbb{R})$ a Boolean algebra of the form $P(Y)$ for some set Y?

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.