COMPLEX ANALYSIS EXAMPLES 3

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These questions are the usual mix not all equally difficult. The first four are maybe tricky, so make sure that you do all the integrals.
I welcome both comments and corrections which can be sent to `m.hyland@dpmms.cam.ac.uk`.

1. Let f be a function analytic on $\mathbb{C}$ apart from a finite number of poles. Show that if there exists k such that $|f(z)| \leq |z|^k$ for all sufficiently large z, then f is a rational function (i.e. the quotient of two polynomials).

2. Suppose that f is an analytic function on $\{z : 0 < |z-a| < R\}$. Show that if the singularity at $z = a$ is not removable then $\exp f(z)$ has an essential singularity at $z = a$. Deduce that if there exists M such that $\Re f(z) < M$ for $0 < |z-a| < R$ then f has a removable singularity at $z = a$.

3. Suppose that $f : D(0,1) \to D(0,1)$ is analytic with $f(0) = 0$.
(i) Show that the function $g(z) = f(z)/z$ has a removable singularity at 0. Use the maximum modulus principle to deduce that $|f(z)| \leq |z|$ for all $z \in D(0,1)$.
[Careful. You are not told and do not need to know anything about behaviour at the boundary.]
(ii) Suppose that $|f(a)| = |a|$ for some $a \neq 0$. Show that $f(z) = \omega z$ for some ω with $|\omega| = 1$.

4. Suppose that f_n are analytic, and that $f_n \to f$ locally uniformly and with f not constant. Show that for any $a \in D$ there is $N(a) \in \mathbb{N}$ and a sequence a_n for $n \geq N(a)$ with $a_n \to a$ as $n \to \infty$ and with $f_n(a_n) = f(a)$.
[You use some later theorem but this can be attempted quite early in the course.]

5. Using the residue theorem establish the following.

$$\text{(i)} \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 10x^2 + 9} dx = \frac{\pi}{4}; \qquad \text{(ii)} \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = \frac{\pi}{\sqrt{2}};$$

$$\text{(iii)} \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 1} dx = \frac{\pi}{\sqrt{2}}; \qquad \text{(iv)} \int_{-\infty}^{\infty} \frac{dx}{x^6 + 1} = \frac{2\pi}{3}.$$

[How many of these integrals can you calculate by standard real variable techniques?]

6. For $a,\ b > 0$ and $a \neq b$ evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx$. Also evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)^2} dx$. Can the latter be deduced from the former by letting $b \to a$?

7. For $-1 < \alpha < 1$, and $\alpha \neq 0$, compute $\displaystyle\int_0^{\infty} \frac{x^\alpha dx}{1 + x + x^2}$. Letting $\alpha \to 0$ and recalculate $\displaystyle\int_0^{\infty} \frac{dx}{1 + x + x^2}$. (You should get the same answer viz $2\pi/3\sqrt{3}$ as in lectures.)

8. Let $a > 0$. For $\omega \in \mathbb{R}$ evaluate the following integrals.

$$\text{(a)}\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{-i\omega x} dx \qquad \text{(b)}\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin x}{x} e^{-i\omega x} dx. \qquad \text{(c)}\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{x^2 + a^2} dx.$$

9. Compute the following integrals.

$$\int_{-1}^{1} x^2(1-x^2)^{1/2}dx; \qquad \int_{-1}^{1} \frac{dx}{(2-x)(1-x^2)^{1/2}}.$$

[Explain how you got the sign right in each case?]

10. (i) For a positive integer N let γ_N be the square contour with vertices $(\pm 1 \pm i)(N+1/2)$. Show that there exists a constant $C > 0$ such that $|\cot \pi z| < C$ on every γ_N.

(ii) By integrating $\dfrac{\pi \cot \pi z}{z^2+1}$, show that $\displaystyle\sum_0^\infty \frac{1}{n^2+1} = \frac{1+\pi\coth\pi}{2}$.

(iii) Evaluate $\displaystyle\sum_0^\infty \frac{(-1)^n}{n^2+1}$.

11. Let $f : D \to \mathbb{C}$ be analytic and take $a \in D$ with $f'(a) \neq 0$. Show that for $r > 0$ sufficiently small the formula

$$g(w) = \frac{1}{2\pi i}\int_{|z-a|=r} z\frac{f'(z)}{f(z)-w}dz$$

defines an analytic function in some neighbourhood of $f(a)$ which is inverse to f.

12. (a) Show that $z^4 + z + 1$ has one zero in each quadrant. Show that all roots lie inside the circle $|z| = 3/2$.

(b) How many zeros does $z^4 + 12z + 1$ have in the annulus $2 < |z| < 3$? Are they distinct? Can you determine in which quadrants they lie?

(c) Find an annulus centre 0 in which $z^4 + 26z + 4$ has exactly three roots. Can you determine in which quadrants they lie?

13. Consider the polynomials

(a) $p(z) = z^4 + z^3 + 2z^2 + 5z + 2$;
(b) $p(z) = z^4 + z^3 + 2z^2 + 5z + 3$;
(c) $p(z) = z^4 + z^3 + 2z^2 + 5z + 4$.

In each case determine whether $p(z)$ has real roots and determine in which quadrants the non-real roots lie.

14 Establish the following refinement of the Fundamental Theorem of Algebra.
Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial of degree n, and $A = \max\{|a_i| : 0 \leq i \leq n-1\}$. Then $p(z)$ has n roots counting multiplicities in the disc $|z| < A + 1$.

15. Prove that $z \sin z = 1$ has only real solutions. [How many real roots are there in the interval $[-(n+1/2)\pi, (n+1/2)\pi]$? How many roots are there in the disc $|z| < (n+1/2)\pi$?]

16. Show that if $|a| > e$, then $az^n = e^z$ has n distinct solutions in the unit disc. Find an upper bound r such that if $|a| < r$ then $az^n = e^z$ has no solutions in the unit disc. Can you say anything when $r < |a| < e$?

17. Prove the following strengthened form of Rouché's Theorem.
Suppose that the analytic functions f and g are such that $|g| < |f| + |f+g|$ on a simple closed curve γ. Then f and $f+g$ have the same number of zeros inside γ.

Finally an additional question to think about. Perhaps for once you really will use the Jordan Curve Theorem?

18. Suppose that γ is a simple closed curve contained (with its interior) in a domain D. Suppose that $f : D \to \mathbb{C}$ is an analytic function which takes no value more than once on γ. Show that f takes no value more than once inside γ.