Linear Algebra: Example Sheet 4

1. Which of the following symmetric matrices are congruent to the identity matrix (a) over $\mathbb{Q}$, (b) over $\mathbb{R}$ and (c) over $\mathbb{C}$?
$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$
[Try to get away with the minimum calculation.]

2. Find the rank and signature of the following quadratic forms over $\mathbb{R}$.
$$x^2 + y^2 + z^2 - 2xz - 2yz, \quad x^2 + 2y^2 - 2z^2 - 4xy - 4yz, \quad 16xy - z^2, \quad 2xy + 2yz + 2zx.$$

3. Show that the map $A \to \operatorname{tr}(A^2)$ is a quadratic form on the space $\mathcal{M}_n$ of all $n \times n$ real matrices. Find its rank and signature.

4. Let $f_1, \cdots, f_t, f_{t+1}, \cdots, f_{t+u}$ be linear functionals on the finite dimensional vector space V. Show that $Q(\mathbf{x}) = f_1(\mathbf{x})^2 + \cdots + f_t(\mathbf{x})^2 - f_{t+1}(\mathbf{x})^2 - \cdots - f_{t+u}(\mathbf{x})^2$ is a quadratic form on V. Suppose Q has rank $p + q$ and signature $p - q$. Show that $p \le t$ and $q \le u$.

5. Suppose that Q is a non-singular quadratic form on V of dimension $2m$. Suppose that Q vanishes on $U \le V$ with $\dim U = m$. Establish the following.
(i) We can write $V = U \oplus W$ with q also vanishing on W.
(ii) There is a basis with respect to which Q has the form $x_1x_2 + x_3x_4 + \cdots x_{2m-1}x_{2m}$.

6. Let J be an $m \times m$ real matrix with $J^2 = -I$. Show that the dimension m is even.
Set $m = 2n$. Show that there is an invertible matrix P such that $P^{-1}JP = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.
Find the dimension of the space of all matrices with $A^tJ + JA = 0$.

7. Show that the quadratic form $2(x^2 + y^2 + z^2 - xy - yz + zx)$ is positive definite. *Write down* an orthonormal basis for the corresponding inner product on $\mathbb{R}^3$. Compute the basis obtained by applying the Gram-Schmidt process to the standard basis.

8. Let $W \le V$ with V an inner product space. Show that the orthogonal projection onto W is a self-adjoint projection. Conversely show that any self-adjoint projection is orthogonal projection onto its image.

9. Let S be a real symmetric matrix with $S^k = I$ for some $k \ge 1$. Show that $S^2 = I$.

10. Let P_n be the ($n + 1$-dimensional) space of real polynomials of degree $\le n$. Define
$$\langle f, g\rangle = \int_{-1}^{+1} f(t)g(t)dt\,.$$
Show that $\langle\ ,\ \rangle$ is an inner product on P_n and that the endomorphism $\alpha : P_n \to P_n$ defined by
$$\alpha(f)(t) = (1 - t^2)f''(t) - 2tf'(t)$$
is self-adjoint. What are the eigenvalues of α?
Now (for want of time in the course) we cheat a bit! Let $s_k \in P_n$ be defined by $s_k(t) = \frac{d^k}{dt^k}(1 - t^2)^k$. Prove the following.
(i) For $i \ne j$, $\langle s_i, s_j\rangle = 0$.
(ii) $s_0, \ldots, s_n$ forms a basis for for P_n.
(iii) For all $1 \le k \le n$, s_k spans the orthogonal complement of P_{k-1} in P_k.
(iv) s_k is an eigenvector of α. (Give its eigenvalue.)
What is the relation between the s_k and the result of applying Gram-Schmidt to the sequence 1, x, x^2, x^3 and so on? (Calculate the first few terms?)

11. An endomorphism α of a finite-dimensional inner product space V is *positive semi-definite* if and only if it is self-adjoint and satisfies $\langle \alpha(\mathbf{x}, \mathbf{x}\rangle \geq 0$ for all $\mathbf{x} \in V$. Prove that a positive semi-definite endomorphism has a unique positive semi-definite square root. (How many square roots do you think it has?)

12. Find a linear transformation which reduces the pair of real quadratic forms

$$2x^2 + 3y^2 + 3z^2 - 2yz, \qquad x^2 + 3y^2 + 3z^2 + 6xy + 2yz - 6zx$$

to the forms

$$X^2 + Y^2 + Z^2, \qquad \lambda X^2 + \mu Y^2 + \nu Z^2$$

for some $\lambda, \mu, \nu \in \mathbb{R}$ (which you will inevitably determine).

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13. Find the rank and signature of the form on $\mathbb{R}^n$ with matrix

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & & & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}.$$

14. Let (r, ϕ, θ) be the standard spherical polar coordinates in $\mathbb{R}^3$. Let $\mathbf{u}_r$, $\mathbf{u}_\phi$ and $\mathbf{u}_\theta$ represent the unit vectors in the direction of increasing r, ϕ and θ respectively.
Write down the matrix giving $\mathbf{u}_r$, $\mathbf{u}_\phi$ and $\mathbf{u}_\theta$ in terms of the standard basis $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ for $\mathbb{R}^3$.
What is the inverse of this matrix and why?
What real value must be an eigenvalue of the matrix and why?
(You could check by explicit calculation of the characteristic polynomial that you are right.)

15. Let S be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \dots\ \lambda_n$. Find the eigenvalues of the endomorphism $X \to SX^tS$ of the space $\mathcal{M}_n$ of $n \times n$-matrices.

16. (i) Show that $O_n(\mathbb{R})$ has a normal subgroup $SO_n(\mathbb{R})$ consisiting of the real orthogonal matrices of determinant $+1$.
(ii) Show that the centre of $O_n(\mathbb{R})$ is $\{\pm I\}$.
(iii) Show that $O_n(\mathbb{R})$ is the direct product of $SO_n(\mathbb{R})$ and the centre $\{\pm I\}$ if and only if n is odd.
(iv) Show that if n is even then $O_n(\mathbb{R})$ is not the direct product of $SO_n(\mathbb{R})$ with any normal subgroup.

17. Consider the space P of polynomials in variables x_1, ... , x_n. We have linear operators $\partial_i = \frac{\partial}{\partial x_i}$; so for any polynomial $f(x_1, \dots, x_n) \in P$ we have a corresponding linear operator $\hat{f} = f(\partial_1, \dots, \partial_n)$. Consider

$$\langle f, g\rangle = \hat{f}(g)(\mathbf{0}),$$

that is the result of applying $f(\partial_0, \dots, \partial_n)$ to $g(x_1, \dots, x_n)$ and then evaluating at $(0, \dots, 0)$. Show that $\langle f, g\rangle$ is an inner product on P.
Fix $g \in P$. What is the adjoint of the map $P \to P; h \to gh$?
Now consider the subspaces $P(d)$ of polynomials homogeneous of degree d. Show that the Laplacian $\Delta = \partial_1^2 + \dots + \partial_n^2 : P(d) \to P(d-2)$ is surjective.

18. Let P and Q be 3×3 orthogonal matrices with determinant 1. Show that $\mathrm{r}(P + Q)$ is odd.

19. Let a_1, a_2, ... , a_n be real numbers such that $a_1 + \dots + a_n = 0$ and $a_1^2 + \dots a_n^2 = 1$. What is the maximum value of $a_1a_2 + a_2a_3 + \dots + a_{n-1}a_n + a_na_1$?

20. Prove Hadamard's Inequality: if A is a real $n \times n$ matrix with $|a_{ij}| \leq k$, then

$$|\det A| \leq k^n n^{n/2}.$$

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.