Michaelmas Term 2003 J. M. E. Hyland

Linear Algebra: Example Sheet 2

The first 12 questions cover the course and should ensure good understanding of the course: the remainder do vary in difficulty but cover some instructive points.

1. For what values of a and b does the system of simultaneous linear equations

$$\begin{aligned} x + y + z &= 1 \\ ax + 2y + z &= b \\ a^2x + 4y + z &= b^2 \end{aligned}$$

have (i) a unique solution, (ii) no solution, (iii) many solutions?

2. Let A and B be $n \times n$ matrices over a field $\mathbb{F}$. Show that the $(2n \times 2n)$ matrix

$$C = \begin{pmatrix} I & B \\ -A & O \end{pmatrix} \quad \text{can be transformed into} \quad D = \begin{pmatrix} I & B \\ 0 & AB \end{pmatrix}$$

by elementary row operations. By considering the determinants of C and D, obtain another proof that $\det AB = \det A \det B$.

3. Let C be an $n \times n$ matrix over $\mathbb{C}$, and write $C = A + iB$, where A and B are real $n \times n$ matrices. By considering $\det(A + \lambda B)$ as a function of λ, show that if C is invertible then there exists a real number λ such that $A + \lambda B$ is invertible. Deduce that if two $n \times n$ real matrices P and Q are conjugate when regarded as matrices over $\mathbb{C}$, then they are conjugate as matrices over $\mathbb{R}$.

4. Show that there are no endomorphisms α, β of a finite dimensional vector space V with $\alpha\beta - \beta\alpha = I$, except for the case $\dim V = 0$.
Find endomorphisms of an infinite dimensional vector space V which do satisfy $\alpha\beta - \beta\alpha = I$.

5. Find a basis with respect to which $\begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$ is diagonal. Hence compute the nth power $\begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}^n$.

6. Compute the characteristic polynomials of the matrices

$$\begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Which of the matrices are diagonalizable over $\mathbb{C}$? Which over $\mathbb{R}$?

7. Let α be an endomorphism of a finite dimensional complex vector space. Show that if λ is an eigenvalue for α then λ^2 is an eigenvalue for α^2. Show further that every eigenvalue of α^2 arises in this way.
Are the eigenspaces $\ker(\alpha - \lambda I)$ and $\ker(\alpha^2 - \lambda^2 I)$ necessarily the same?

8. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The second and third matrices commute, so find a basis with respect to which they are both diagonal.

9. Suppose that $\alpha \in \mathcal{L}(V, V)$ is invertible. Describe the characteristic and minimal polynomials and the eigenvalues of α^{-1} in terms of those of α.

10. Find the characteristic polynomial and the algebraic and geometric multiplicities of the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 1 & 3 & -1 & 2 \\ 0 & 0 & -1 & 0 \\ -1 & -2 & 1 & -1 \end{pmatrix}.$$

[Be sensible: little calculation is needed.] Now what is the minimum polynomial?

11. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Show that the characteristic polynomial is $t^3 - 2t + 1$. Hence compute $A^7 - 2A^5 + 2A^4 - 2A^2 + 2A + I$ and A^{-1}.

12. Show that an endomorphism $\alpha : V \to V$ of a finite dimensional complex vector space V has 0 as only eigenvalue if and only if it is *nilpotent*, that is, $\alpha^k = 0$ for some natural number k. Show that the minimum such k is at most $\dim(V)$. What can you say if the only eigenvalue of α is 1?

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13. Suppose that $A : \mathbb{C}^n \to \mathbb{C}^n$ has eigenvalues $\lambda_1, \dots, \lambda_n$. Regard $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as a $2n$-dimensional real vector space, and consider the endomorphism $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. What are the complex eigenvalues of this A?

14. Let A be an $n \times n$ matrix all the entries of which are real. Show that the minimum polynomial of A, over the complex numbers, has real coefficients.

15. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_n & a_0 & a_1 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \dots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix}.$$

Show that the determinant of C is $\det C = \prod_{j=0}^{n} f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.

16. Let A, B be $n \times n$ matrices, where $n \geq 2$. Show that, if A and B are non-singular, then

$$(i)\ \mathrm{adj}(AB) = \mathrm{adj}(B)\mathrm{adj}(A)\,, \quad (ii)\ \det(\mathrm{adj}A) = (\det A)^{n-1}\,, \quad (iii)\ \mathrm{adj}(\mathrm{adj}A) = (\det A)^{n-2}A\,.$$

What happens if A is singular?

Show that the rank of the matrix adj$A$ is $\mathrm{r}(\mathrm{adj}(A)) = \begin{cases} n & \text{if } \mathrm{r}(A) = n; \\ 1 & \text{if } \mathrm{r}(A) = n-1; \\ 0 & \text{if } \mathrm{r}(A) \leq n-2. \end{cases}$

17. (i) An endomorphism $\alpha : V \to V$ of a finite dimensional vector space is *periodic* just when $\alpha^k = I$ for some k. Show that a periodic matrix is diagonalisable over $\mathbb{C}$.
(ii) Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a basis for a vector space V over $\mathbb{C}$. For σ a permutation of $\{1, \dots, n\}$, define $\widehat{\sigma} : V \to V$ by $\widehat{\sigma}(\mathbf{e}_i) = \mathbf{e}_{\sigma(i)}$. What are the eigenvalues of $\widehat{\sigma}$? [Consider the case when σ is a cycle first?]
(iii) Is every periodic endomorphism of the form $\widehat{\sigma}$ for some choice of permutation σ and basis $\mathbf{e}_1, \dots, \mathbf{e}_n$?

18. Let V be a complex vector space with dimension n and let α be an endomorphism of V with $\alpha^{n-1} \neq 0$ but $\alpha^n = 0$. Show that there is a vector $\mathbf{x} \in V$ for which

$$\mathbf{x},\ \alpha(\mathbf{x}),\ \alpha^2(\mathbf{x}),\ \dots,\ \alpha^{n-1}(\mathbf{x})$$

is a basis for V. Give the matrix of α relative to this basis.
Let $p(t) = a_0 + a_1 t + \ldots + a_k t^k$ be a polynomial. What is the matrix for $p(\alpha)$ with respect to the base? What is the minimal polynomial for α? What are the eigenvalues and eigenvectors?
Show that if an endomorphism β of V commutes with α then $\beta = p(\alpha)$ for some polynomial $p(t)$. (It may help to consider $\beta(\mathbf{x})$.)

19. Let $\alpha : V \to V$ be an endomorphism of a finite dimensional vector space V with $\mathrm{tr}(\alpha) = 0$.
(i) Show that, if $\alpha \neq 0$, there is a vector $\mathbf{v}$ with $\mathbf{v}, \alpha(\mathbf{v})$ linearly independent. Deduce that there is a basis for V relative to which α is represented by a matrix A with all of its diagonal entries equal to 0.
(ii) Show that there are endomorphisms β, γ of V with $\alpha = \beta\gamma - \gamma\beta$.

20. (i) Suppose that the endomorphism $A : \mathbb{C}^n \to \mathbb{C}^n$ is nilpotent. Show that $\mathrm{tr}(A) = 0$.
(ii) Suppose $\lambda_1, \dots, \lambda_n$ are such that $\sum \lambda_i^r = 0$ for $1 \leq r \leq n$. Show that the $\lambda_1, \dots, \lambda_n$ are all 0. [This follows trivially from a famous result on symmetric functions, but you can prove it directly.]
(iii) Deduce that if the endomorphism $A : \mathbb{C}^n \to \mathbb{C}^n$ is such that $\mathrm{tr}(A^k) = 0$ for all k then A is nilpotent.

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.