Michaelmas Term 2013 J. M. E. Hyland

Linear Algebra: Example Sheet 1

The first 12 questions cover the relevant part of the course and should ensure good understanding. A few other questions are included in case you have time for them.

1. $\mathbb{R}^{\mathbb{R}}$ is the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$, with addition and scalar multiplication defined pointwise. Which of the following sets of functions form a vector subspace of $\mathbb{R}^{\mathbb{R}}$?
(a) The set C of continuous functions.
(b) The set $\{f \in C : |f(t)| \leq 1 \text{ for all } t \in [0,1]\}$.
(c) The set $\{f \in C : f(t) \to 0 \text{ as } t \to \infty\}$.
(d) The set $\{f \in C : f(t) \to 1 \text{ as } t \to \infty\}$.
(e) The set $\{f \in C : |f(t)| \to \infty \text{ as } |t| \to \infty\}$.
(f) The set of solutions of the differential equation $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4 x(t) = 0$.
(g) The set of solutions of $\ddot{x}(t) + (t^2 - 3)\dot{x}(t) + t^4 x(t) = \sin t$.
(h) The set of solutions of $(\dot{x}(t))^2 - x(t) = 0$.
(i) The set of solutions of $(\ddot{x}(t))^4 + (x(t))^2 = 0$.

2. (i) Suppose that T and U are subspaces of the vector space V. Show that $T \cup U$ is also a subspace of V if and only if either $T \leq U$ or $U \leq T$.
(ii) Let T, U and W be subspaces of V. Give explicit counter-examples to the following statements.

   (a) $T + (U \cap W) = (T + U) \cap (T + W)$. (b) $(T + U) \cap W = (T \cap W) + (U \cap W)$.

   Show that each of these equalities can be replaced by a valid inclusion of one side in the other.

3. If α and β are linear maps from U to V, show that $\alpha + \beta$ is linear. Give explicit counter-examples to the following statements.

   (a) $\mathrm{Im}(\alpha + \beta) = \mathrm{Im}\alpha + \mathrm{Im}\beta$: (b) $\ker(\alpha + \beta) = \ker\alpha \cap \ker\beta$.

   Show that each of these equalities can be replaced by a valid inclusion of one side in the other.

4. Suppose that $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a base for V. Which of the following are also bases?
(a) $\{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n\}$.
(b) $\{\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} + \mathbf{e}_n, \mathbf{e}_n + \mathbf{e}_1\}$.
(c) $\{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} - \mathbf{e}_n, \mathbf{e}_n\}$.
(d) $\{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} - \mathbf{e}_n, \mathbf{e}_n - \mathbf{e}_1\}$.
(e) $\{\mathbf{e}_1 - \mathbf{e}_n, \mathbf{e}_2 + \mathbf{e}_{n-1}, \ldots, \mathbf{e}_n + (-1)^n \mathbf{e}_1\}$.

5. For each of the following pairs of vector spaces (V, W) over $\mathbb{R}$, either give an isomorphism $V \to W$ or show that no such isomorphism can exist. (Here P denotes the space of polynomial functions $\mathbb{R} \to \mathbb{R}$, and $C[a, b]$ denotes the space of continuous functions defined on the closed interval $[a, b]$.)
(a) $V = \mathbb{R}^4$, $W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 0\}$.
(b) $V = \mathbb{R}^5$, $W = \{p \in P : \deg\ p \leq 5\}$.
(c) $V = C[0,1]$, $W = C[-1,1]$.
(d) $V = C[0,1]$, $W = \{f \in C[0,1] : f(0) = 0, f \text{ continuously differentiable }\}$.
(e) $V = \mathbb{R}^2$, $W = \{\text{solutions of } \ddot{x}(t) + x(t) = 0\}$.
(f) $V = \mathbb{R}^4$, $W = C[0,1]$.
(g) $V = P$, $W = \mathbb{R}^{\mathbb{N}}$.

6. Let

$$U = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_3 + x_4 = 0,\ 2x_1 + 2x_2 + x_5 = 0\}, \quad W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + x_5 = 0,\ x_2 = x_3 = x_4\}.$$

   Find bases for U and W containing a basis for $U \cap W$ as a subset. Give a basis for $U + W$ and show that

$$U + W = \{\mathbf{x} \in \mathbb{R}^5 : x_1 + 2x_2 + x_5 = x_3 + x_4\}.$$

7. The vector space F^n has a standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of unit vectors. Let W be a subspace of F^n. Show that there is a finite subset I of $\{1, 2, \ldots, n\}$ for which the subspace $U = \langle\{\mathbf{e}_i : i \in I\}\rangle$ is a complementary subspace to W in F^n.

8. Let $U_1, \ldots, U_r$ be subspaces of a vector space V, and suppose that for each i, B_i is a basis for U_i. Show that the following conditions are equivalent.

   (i) $U = \sum_i U_i$ is a direct sum, that is, every element of U can be uniquely expressed as a sum $\sum_i \mathbf{u}_i$ with $\mathbf{u}_i \in U_i$;
   (ii) For each j, $U_j \cap \sum_{i \neq j} U_i = \{0\}$.
   (iii) The B_i are pairwise disjoint and their union B is a basis for $U = \sum_i U_i$.
   Give an example where $U_i \cap U_j = \{0\}$ for all $i \neq j$, and yet $U = \sum_i U_i$ is not a direct sum.

9. Let $\alpha : U \to V$ be a linear map between two finite dimensional vector spaces and let W be a vector subspace of U. Show that the restriction of α to W is a linear map $\alpha|_W : W \to V$ which satisfies

   $$\mathrm{r}(\alpha) \geq \mathrm{r}(\alpha|_W) \geq \mathrm{r}(\alpha) - \dim(U) + \dim(W) \ .$$

   Give examples to show that either of the two inequalities can be an equality.

10. Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $\alpha : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Find the matrix representing α relative to the base $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for both the domain and the range.
    Write down bases for the domain and range with respect to which the matrix of α is the identity.

11. Let Y and Z be subspaces of the finite dimensional vector spaces V and W respectively. Show that $R = \{\theta \in \mathcal{L}(V, W) : \theta(\mathbf{x}) \in Z \text{ for all } \mathbf{x} \in Y\}$ is a subspace of $\mathcal{L}(V, W)$. What is the dimension of R?

12. (i) Let $\alpha : V \to V$ be an endomorphism of a finite dimensional vector space V. Set $r_i = \mathrm{r}(\alpha^i)$. Show that $r_i \geq r_{i+1}$ and that $(r_i - r_{i+1}) \geq (r_{i+1} - r_{i+2})$. Deduce that if $r_k = r_{k+1}$ for some $k \geq 0$, then r_j is constant for all $j \geq k$.
    (ii) Suppose that $\dim(V) = 5$, $\alpha^3 = 0$, but $\alpha^2 \neq 0$. What possibilities are there for $\mathrm{r}(\alpha)$ and $\mathrm{r}(\alpha^2)$?

---

13. Let T, U and W be subspaces of a vector space. Show that if $T \leq W$, then

    $$(T + U) \cap W = (T \cap W) + (U \cap W) \, .$$

    Deduce that in general one has $T \cap (U + (T \cap W)) = (T \cap U) + (T \cap W)$.

14. Let W be a subspace of a finite dimensional vector space V. Show that both W and V/W are finite dimensional and that $\dim V = \dim W + \dim V/W$.

15. Suppose X and Y are linearly independent subsets of a vector space V; no member of X is expressible as a linear combination of members of Y, and no member of Y is expressible as a linear combination of members of X. Is the set $X \cup Y$ necessarily linearly independent? Give a proof or counterexample.

16. (Another version of the Exchange Lemma.) Let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_s\}$ be linearly independent subsets of a vector space V, and suppose $r \leq s$. Show that it is possible to choose distinct indices $i_1, i_2, \ldots, i_r$ from $\{1, 2, \ldots, s\}$ such that, if we delete each $\mathbf{y}_{i_j}$ from Y and replace it by $\mathbf{x}_j$, the resulting set is still linearly independent. Deduce that any two maximal linearly independent subsets of a finite dimensional vector space have the same size.

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.