Which Spectrum?

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Outline

Motivation

In general:

Geometric ergodicity $\Leftrightarrow$ spectral gap in L_∞^V

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Under *reversibility*:

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Under *reversibility*:

Geometric ergodicity $\Leftrightarrow$ spectral gap in L_2

In the absence of reversibility

Geometric ergodicity $\Leftarrow$ spectral gap in L_2 (explicit)

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Geometric ergodicity $\not\Rightarrow$ spectral gap in L_2 (example)

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Geometric ergodicity $\not\Rightarrow$ spectral gap in L_2 (example)

Convergence rates

Under reversibility:

TV finite-n bound

Without reversibility:

Asymptotic V-norm bound

The Setting

$\{X_n\}$	Markov chain with general state space $(\Sigma, \mathcal{S})$
$X_0 = x \in \Sigma$	initial state
$P(x, dy)$	transition kernel

$$P(x, A) := \mathsf{P}_x\{X_1 \in A\} := \Pr\{X_n \in A | X_{n-1} = x\}$$

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ψ-irreducibility and aperiodicity

Assume that there exists σ-finite measure ψ on $(\Sigma, \mathcal{S})$
such that $\qquad P^n(x, A) > 0 \qquad$ eventually
for any $x \in \Sigma$ and any $A \in \mathcal{S}$ with $\psi(A) > 0$

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Recall

Any kernel $Q(x, dy)$ acts of functions $F : \Sigma \to \mathbb{R}$
and measures μ on $(\Sigma, \mathcal{S})$ as a linear operator:

$$QF(x) = \int_\Sigma Q(x, dy) F(y) \qquad \mu Q(A) = \int_\Sigma \mu(dx) Q(x, A)$$

Geometric Ergodicity (GE) Equivalent Conditions

⇝ There is an invariant measure π and functions $\rho : \Sigma \to (0,1)$, $C : \Sigma \to [1,\infty)$:

$$\|\boldsymbol{P^n(x,\cdot) - \pi}\|_{\mathbf{TV}} \leq \boldsymbol{C(x)\rho(x)^n} \quad \boldsymbol{n \geq 0, \ \pi} - \text{a.s.}$$

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- There is an invariant measure π
  constants $\rho \in (0,1)$, $B < \infty$ and a π-a.s. finite $V : \Sigma \to [1,\infty]$:

$$\|\boldsymbol{P^n(x,\cdot) - \pi}\|_{\boldsymbol{V}} \leq \boldsymbol{BV(x)\rho^n} \quad \boldsymbol{n \geq 0,\ \pi -} \text{a.s.}$$

where $\quad \|F\|_V := \sup_{x\in\Sigma} \frac{|F(x)|}{V(x)} \qquad \|\mu\|_V := \sup_{F:\|F\|_V<\infty} \left| \int F d\mu \right|$

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- **Lyapunov condition (V4)**
  There exist $V : \Sigma \to [1,\infty)$, $\delta > 0$, $b < \infty$ and a "small" $C \subset \Sigma$:

$$\boldsymbol{PV(x) \le (1-\delta)V(x) + b\mathbb{I}_C}$$

GE and the L_∞^V Spectrum

Proposition 1: **Geometric ergodicity $\Leftrightarrow$ spectral gap in L_∞^V**
[∼K-Meyn 2003]

Suppose the chain $\{X_n\}$ is ψ-irreducible and aperiodic.

Then it is GE iff P admits a spectral gap in

$$L_\infty^V := \{F : \Sigma \to \mathbb{R} \text{ s.t. } \|F\|_V < \infty\}$$

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Recall

A set $C \subset \Sigma$ is *small* if there exist $n \geq 1$, $\epsilon > 0$ and a probability measure ν on $(\Sigma, \mathcal{S})$ such that $P^n(x, A) \geq \epsilon \mathbb{I}_C(x)\nu(A)$ for all $x \in \Sigma,\ A \in \mathcal{S}$

The *spectrum* $S(P)$ of $P : L_\infty^V \to L_\infty^V$ is the set of $\lambda \in \mathbb{C}$ s.t. $(I - \lambda P)^{-1} : L_\infty^V \to L_\infty^V$ does *not* exist

$P : L_\infty^V \to L_\infty^V$ admits a *spectral gap* if $S(P) \cap \{z \in \mathbb{C} \ : \ |z| \geq 1 - \epsilon\}$ contains only poles of finite multiplicity for some $\epsilon > 0$

Proof ideas $(\Rightarrow)$

Consider the *potential operator*

$$U_z := [Iz - (P - \mathbb{I}_C \otimes \nu)]^{-1}, \quad z \in \mathbb{C}$$

Iterating the contraction provided (V4) gives a bound on $|||U_z|||_V$ for $z \sim 1$

Use U_z to check that $f_0 \equiv 1$ is an eigenfunction
corresponding to $\lambda_0 = 1$

Using an operator-inversion formula *a la* Nummelin

$$[Iz - P]^{-1} = [Iz - (P - \mathbb{I}_C \otimes \nu)]^{-1} \left(I + \frac{1}{1-\kappa} \mathbb{I}_C \otimes \nu \right)$$

show $\lambda = 1$ is maximal, isolated, and non-repeated

$$\kappa = \nu[Iz - (P - \mathbb{I}_C \otimes \nu)]^{-1} \mathbb{I}_C$$

□

GE, Reversibility and the $L_2(\pi)$ Spectrum

Proposition 2: Under reversibility: GE $\Leftrightarrow$ spectral gap in L_2
[Roberts-Rosenthal 1997] [Roberts-Tweedie 2001] [K-Meyn 2003]

Suppose the chain $\{X_n\}$ is reversible, ψ-irreducible and aperiodic.
Then it is GE iff P admits a spectral gap in $L_2(\pi)$

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Proof

Analogous definitions, proof outline similar to Proposition 1

Big difference:

In the Hilbert space setting, the spectral gap is simply

$$1 - \sup\left\{\frac{\|\nu P\|_2}{\|\nu\|_2} \ : \ \nu \text{ s.t. } \nu(\Sigma) = 1, \ \|\nu\|_2 \neq 0\right\}$$

where $\|\nu\|_2 := \|d\nu/d\pi\|_2$ $\square$

L_2 Spectral Gap Always Implies GE

Theorem 1

Suppose the chain $\{X_n\}$ is ψ-irreducible and aperiodic and that P admits a spectral gap in L_2

Then the chain is geometrically ergodic
[w.r.t. so some Lyapunov function V]

L_2 Spectral Gap Always Implies GE

Theorem 1

Suppose the chain $\{X_n\}$ is ψ-irreducible and aperiodic and that P admits a spectral gap in L_2

Then for any $h \in L_2(\pi)$ there is a $V_h \in L_1(\pi)$ s.t.:

- $\rightsquigarrow$ (V4) holds w.r.t. V_h
- $\rightsquigarrow$ $h \in L_\infty^{V_h}$

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Proof

Prove "soft" GE

Get explicit exponential bounds on explicit Kendall sets

Let

$$V_h(x) := E_x\Big[\sum_{n=0}^{\sigma_C} \Big(1 + |h(X(x))|\Big) \exp\{\tfrac{1}{2}\theta n\}\Big]$$

□

L_2 Spectral Gap $\not\Leftarrow$ GE

Theorem 2

There is a (non-reversible) ψ-irreducible and aperiodic chain $\{X_n\}$ on a countable state space Σ, which is geometrically ergodic but its transition kernel P does *not* admit a spectral gap in L_2

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Proof

Start with an example of Häggström or of Bradley:
GE chain $\{X_n\}$ but CLT fails for some $G \in L_2$

Spectral gap exists
$\Rightarrow$ autocorrelation function of $\{G(X_n)\}$ decays exponentially
$\Rightarrow$ normalized partial sums of $\{G(X_n)\}$ bdd in L_2
$\Rightarrow$ CLT $\Rightarrow$ contradiction $\square$

Why do we care?

Theorem 3. [Roberts-Rosenthal 1997]

Suppose the chain $\{X_n\}$ is reversible, ψ-irreducible and aperiodic

If P admits a spectral gap $\delta_2 > 0$ in L_2

Then for any $X_0 \sim \mu$: $\quad \|\mu P^n - \pi\|_{\mathrm{TV}} \leq \|\mu - \pi\|_2 (1 - \delta_2)^n$

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Theorem 4.

Suppose the chain $\{X_n\}$ is ψ-irreducible and aperiodic

If P admits a spectral gap $\delta_V > 0$ in L_∞^V

Then for π-a.e. x:

$$\lim_{n\to\infty} \frac{1}{n} \log \|P^n(x,\cdot) - \pi\|_V = \log(1-\delta_V)$$

In fact:

$$\lim_{n\to\infty} \frac{1}{n} \log \left(\sup_{x\in \mathsf{X},\ \|F\|_V=1} \frac{|P^n F(x) - \int F\, d\pi|}{V(x)} \right) = \log(1-\delta_V)$$