## Intersection Theory Sheet I

(1) Let $X, Y$ be schemes. Consider the group homomorphism $\times : Z_\star X \otimes Z_\star Y \to Z_\star(X \times Y)$ given by $[V] \otimes [W] \mapsto [V \times W]$ for subvarieties $V, W$ and extending linearly[1].

(a) Let $f : X' \to X, g : Y' \to Y$ be morphisms of schemes, and let $f \times g : X' \times Y' \to X \times Y$ be the induced morphism on products. Show that if $f, g$ are both flat (resp. proper), then $f \times g$ is flat (resp. proper), then $f \times g$ is flat (resp. proper).
(Hint: express $f \times g$ as a composition of two morphisms.)

(b) Show that when $f : X' \to X, g : Y' \to Y$ is flat,

$$(f \times g)^{-1}(\alpha \times \beta) = [f^{-1}(\alpha)] \times [f^{-1}(\beta)] \in Z_\star(X' \times Y'),$$

(c) Show that when $f : X' \to X, g : Y' \to Y$ is proper, we have

$$(f \times g)_*(\alpha \times \beta) = [f_*(\alpha)] \times [f_*(\beta)] \in Z_\star(X \times Y).$$

(d) Prove that the construction induces a well-defined homomorphism

$$\times : A_\star(X) \otimes A_\star(Y) \to A_\star(X \times Y).$$

(Hint: apply the previous items to simplify the proof.)

(2) Let $X$ be a scheme with a cellular decomposition. Prove that for all schemes $Y$, the map constructed in the previous question

$$A_\star(X) \otimes A_\star(Y) \to A_\star(X \times Y)$$

is surjective.

(3) Prove that the Grassmannian $\mathrm{Gr}(d, V)$ is irreducible.

(4) Let $X \subset \mathbb{P}^2$ be a cubic curve with a unique singular point $p$. This exercise calculates the Picard group of $X$. In the following, you may use the following facts:

- every Cartier divisor on $X$ is linearly equivalent to a Cartier divisor such that for some open $p \in U \subset X$, the section $s_U \in \Gamma(U, \mathscr{K}^\star)$ lies in $\Gamma(U, \mathscr{O}^\star)$;
- on a normal scheme $X$, every Weil divisor is Cartier.

(a) Recall the degree function $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$. Prove that the degree function is surjective.

(b) Suppose there is a regular point $q_0 \in X$ which is an inflection point[2]. Let $\mathrm{Pic}^0(X)$ be the kernel of $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$. Prove that the map (a priori of sets) $X \setminus p \to \mathrm{Pic}^0(X)$ given by $q \mapsto q - q_0$ defines a bijection.
(Hint: analogous to the case of a smooth elliptic curve, you may want to construct a group operation on $X \setminus p$. You may use the fact that $p$ does not lie the line through any two points on $X \setminus p$.)

(5) Let $X$ be an $n$-dimensional variety. Recall that there exists a homomorphism $\mathrm{Pic}(X) \to A_{k-1}(X)$.

(a) Let $X$ be the projective plane curve over $\mathbb{C}$ defined by the equation $y^2 z = x^3$. Prove that $A_0 X \cong \mathbb{Z}$, and, using the previous exercise, prove that the kernel of $\mathrm{Pic}(X) \to A_0(X)$ is isomorphic to $\mathbb{C}$ as an abelian group under addition.

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[1] Notice that when $k$ is not algebraically closed, $V \times W$ might not be irreducible, in which case the item $[V \times W]$ is the sum of its irreducible components, weighted by multiplicities.

[2] Namely, the tangent line $L$ of $q_0$ satisfies that $i(q_0, L \cdot X) = 3$.

(b) Let $X$ be the projective plane curve over $\mathbb{C}$ defined by the equation $y^2z = x^2z + x^3$. Prove that $A_0X \cong \mathbb{Z}$ and that the kernel of $\mathrm{Pic}(X) \to A_0(X)$ is isomorphic to $\mathbb{C}^\star$ as an abelian group under multiplication.

(c) Let $X$ be the surface in $\mathbb{A}^3$ defined by $z^2 = xy$. Explain why the subscheme defined by $x = z = 0$ defines a Weil divisor but not a Cartier divisor. Show that $\mathrm{Pic}(X) = 0$ and $A_1(X) = \mathbb{Z}/2\mathbb{Z}$.

(6) Let $X \subset \mathbb{P}^n$ be a closed subscheme of dimension $\leq k$, and let $\mathscr{O}_X(i)$ be the restriction of the line bundle $\mathscr{O}_{\mathbb{P}^n}(i)$. You may assume the fact that the function $t \mapsto H^0(X, \mathscr{O}_X(t))$ is a polynomial function for $t \gg 0$ of degree $\leq k$, called the *Hilbert polynomial* of $(X, \mathscr{O}_X(1))$. Define $d_k(X)$ as the coefficient of $t^k/k!$ of this polynomial.

(a) Let $H = \mathbb{V}(f) \subset \mathbb{P}^n$ be a degree $m$ hypersurface not containing $X$, and let $X \cap H \subset X$ be the scheme-theoretic intersection. We use $\mathscr{O}_{X\cap H}(i)$ to denote the restriction of the line bundle $\mathscr{O}_X(i)$. Recall the short exact sequence

$$0 \to \mathscr{O}_X(-m) \xrightarrow{f} \mathscr{O}_X \to \mathscr{O}_{X\cap H} \to 0.$$

Using this short exact sequence, prove that for $t \gg 0$, there is a short exact sequence

$$0 \to H^0(X, \mathscr{O}_X(t-m)) \to H^0(X, \mathscr{O}_X(t)) \to H^0(X \cap H, \mathscr{O}_{X\cap H}(t-m)) \to 0.$$

Deduce that $d_k(X) = md_{k-1}(X \cap H)$.

(b) For any subvariety $X \subset \mathbb{P}^n$, prove that $d_k(X) = \int_{\mathbb{P}^n} c_1(\mathscr{O}_{\mathbb{P}^n}(1))^k \cap [X]$.
(You may assume that for any $0 < i \leq n$, there exists some codimension-$i$ linear subspace $L_i \subset \mathbb{P}^n$ such that $X \not\subset L_i$ and $X \cap L_i$ is irreducible.)

## Intersection Theory Sheet II

(1) Let $X$ be a scheme and let $E$ be a rank $r$ vector bundle on $X$. Recall that there exists some pullback $f : X' \to X$ such that $f^*E$ splits into a direct sum of line bundles, and their first Chern classes $\{\alpha_1, \ldots, \alpha_r\}$ are called the *Chern roots* of $E$, so that $c_t(f^*E) = \prod_{i=1}^r (1 + \alpha_i t)$. We calculate the Chern polynomial of $\wedge^p E$ as follows:

(a) Suppose $0 \to L \to E \to E' \to 0$ is a short exact sequence of vector bundles with $L$ having rank 1, prove that there is a short exact sequence

$$0 \to \wedge^{p-1} E' \otimes L \to \wedge^p E \to \wedge^p E' \to 0.$$

(b) Using the above result and the fact that $f^*(\wedge^p E) = \wedge^p f^* E$, prove that

$$c_t(f^*(\wedge^p E)) = \prod_{i_1 < \cdots < i_p} (1 + (\alpha_{i_1} + \cdots + \alpha_{i_p})t).$$

(2) Find $X \subseteq \mathbb{P}^n$ and $X \subseteq \mathbb{P}^m$ such that $\deg_{\mathbb{P}^n}(X) \neq \deg_{\mathbb{P}^m}(X)$.

(3) The following exercises help familiarise with regular embeddings.

(a) Find an example of a regular embedding $i : X \to Z$ and a morphism $Y \to Z$ such that the base change $X \times_Z Y \to Y$ is no longer a regular embedding.

(b) Let $i : Z \to Y$ and $j : Y \to X$ be closed embeddings. Pick $z \in Z$ and let $y = i(z)$ and $x = j(y)$. Prove that the following are equivalent:

(i) $i$ is a regular embedding in a neighbourhood of $z$ and $j$ is a regular embedding in a neighbourhood of $y$,

(ii) $i$ and $j \circ i$ are regular embeddings in a neighbourhood of $z$,

(iii) $j \circ i$ is a regular embedding in a neighbourhood of $z$ and that: letting $I = \ker(\mathscr{O}_{X,x} \to \mathscr{O}_{Y,y})$ and $J = \ker(\mathscr{O}_{X,x} \to \mathscr{O}_{Z,z})$, we have that $0 \to I/IJ \to J/J^2 \to J/(I + J^2) \to 0$ is split exact as a sequence of $\mathscr{O}_{X,x}/J$-modules.

(c) Let $i : Z \to X$ be a closed embedding. If $Z$ and $X$ are regular schemes, prove that $i$ is a regular embedding.

(4) Let $X$ be an $n$-dimensional non-singular subvariety of $\mathbb{P}^m$ cut out by some homogeneous equations $f_1, \ldots, f_r$. At $p \in X$, we denote $T_p X \subset \mathbb{P}^m$ as the tangent space of $X$ at $p$, as a projective linear subspace in $\mathbb{P}^m$. Let $\check{\mathbb{P}}^m$ be the projective space of hyperplanes in $\mathbb{P}^m$, and let $\tilde{X} = \{(p, H) \in X \times \check{\mathbb{P}}^m \mid T_p X \subset H\}$. The reduced image $X^\vee$ of the projection map $f : \tilde{X} \to \check{\mathbb{P}}^m$ is known as the *dual variety* of $X$.

(a) Let $N$ be the normal bundle of $X \subset \mathbb{P}^m$, and let $N^\vee$ be the dual of $N$. Prove that $\tilde{X} \to X$ is isomorphic to $p : \mathbb{P}N^\vee \to X$.

(b) Prove that $f^*\mathscr{O}_{\check{\mathbb{P}}^m}(1) = \mathscr{O}_{N^\vee}(1) \otimes \pi^*\mathscr{O}_X(-1)$, where recall that $\mathscr{O}_{N^\vee}(1)$ is dual to the tautological subbundle $\mathscr{O}_{N^\vee}(-1) \to p^*N^\vee$ on $\mathbb{P}N^\vee$.

(c) Let $h = c_1(\mathscr{O}_X(1))$. Prove that

$$\deg f_*[\tilde{X}] = (-1)^n \int_X \frac{c(T_X)}{(1+h)^2}, \text{ and } \deg X^\vee = \frac{(-1)^n}{\deg(\tilde{X}/X^\vee)} \int_X \frac{c(T_X)}{(1+h)^2}.$$

(d) Let $X$ be a non-singular plane curve of degree $d$. Embed $X$ in $\mathbb{P}^m$, $m = \binom{r+2}{2} - 1$ by the $r$-fold Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^m$. The dual variety $X^\vee$ to $X$ in $\mathbb{P}^m$ is the variety of plane curves of degree $r$ which are tangent to $X$. Using the above formula, prove that the degree of $X^\vee$ is $d(d + 2r - 3)$.

## Intersection Theory Sheet III

[*Starred questions are optional and non-examinable.*]

(1) Let $\sigma : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{(r+1)(s+1)-1}$ be the Segre embedding

$$([x_0 : \cdots : x_r], [y_0 : \cdots : y_s]) \mapsto [x_0 y_0 : \cdots : x_i y_j : \cdots : x_r y_s].$$

You may assume the fact that $\sigma$ defines a closed embedding. Calculate $\deg(\sigma(\mathbb{P}^r \times \mathbb{P}^s))$.

(2) Let $V \subset \mathbb{P}^n$ be a codimension $m$ complete intersection with homogenous polynomials $F_1, \ldots, F_m$ of degrees $d_1, \ldots, d_m$ respectively.

(a) Calculate the degree of $V$.

(b) [*Please attempt this part after Koszul complexes are covered in the lectures.*] Calculate $N_{V/\mathbb{P}^n}$ as the restriction of some vector bundle on $\mathbb{P}^n$.

(3) Let $Y$ be a pure-dimensional scheme of dimension $n$ and let $X \subset Y$ be a subscheme such that the irreducible components of $X$, denoted as $X_1, \ldots, X_m$, all have dimension $d$.

(a) Explain why $s(X, Y) \in A_*(X)$ is a sum of classes in degrees $* \leq d = \dim X$, and that its summand in $A_d(X)$ must be equal to $\sum_{i=1}^m (e_X Y)_{X_i} \cdot [X_i]$ for some $(e_X Y)_{X_i} \in \mathbb{Z}$.

(b) Define $(e_X Y)_{X_i}$ as *the multiplicity* of $Y$ along $X$ at $X_i$. Let $p : \mathbb{P}(C_{X_i} Y) \to X$. Prove that

$$(e_X Y)_{X_i} \cdot [X_i] = p_*(c_1(\mathscr{O}(1))^{n-d-1} \cap [\mathbb{P}(C_{X_i} Y)]).$$

(c) Let $V \subset X$ be an irreducible component and let $Y_1, \ldots, Y_r$ be the irreducible components of $Y$ which contain $V$ with geometric multiplicities $m_1, \ldots, m_r$, prove that

$$(e_X Y)_V = \sum_{i=1}^r m_i \cdot (e_{X \cap Y_i} Y_i)_V.$$

(d) Let $f : Y' \to Y$ be a proper surjective morphism of irreducible varieties. Let $V'$ be a subvariety of $Y'$, $V = f(V')$, so that $V$ is irreducible. If $V''$ is an irreducible component of $f^{-1}(V)$, define the *ramification index* of $f$ at $V''$, $e_{V''}(f)$, to be the multiplicity of $Y'$ along $f^{-1}(V)$ at $V''$. If all irreducible components $V''$ of $f^{-1}(V)$ have the same dimension as $V$, prove that

$$\deg(Y'/Y) e_V Y = \sum_{V''} \deg(V''/V) e_{V''}(f).$$

(e)$^\star$ In the following, let the subscheme $X \subset Y$ be a closed point $X = p$. In this exercise, we verify that the above definition of multiplicity agrees with the following algebraic one. Let $\mathscr{O}_{Y,p}$ be the local ring of $p \subset Y$ with maximal ideal $\mathfrak{m}$. Define the Hilbert–Samuel function of $\mathscr{O}_{Y,p}$ as $HS(\mathscr{O}_{Y,p}) : t \mapsto \sum_{i=0}^t \dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1})$.

(i) Let $S_{Y,p}[z] := \left(\bigoplus_{i=0}^\infty \mathfrak{m}^i/\mathfrak{m}^{i+1}\right)[z]$ be the graded $k$-algebra where each $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ receives grading $i$ and $z$ has grading 1. Recall that $\mathrm{Proj}(S_{Y,p}[z])$ is a projective scheme with line bundle $\mathscr{O}(1)$. Prove that the function $HS(\mathscr{O}_{Y,p})$ agrees with the Hilbert function of $(\mathrm{Proj}(S_{Y,p}[z]), \mathscr{O}(1))$.

(ii) Using the results about Hilbert polynomials and degrees in Sheet 1, prove that $HS(\mathscr{O}_{Y,p})$ is a polynomial for $t \gg 0$. Let the leading term of the polynomial be $m_p Y \cdot t^{n-d}/(n-d)!$. Give a formula of $m_p Y$ in terms of $(\mathrm{Proj}(S_{Y,p}[z]), \mathscr{O}(1))$.

(iii) Deduce that $m_p Y$ agrees with $e_p Y$ as defined above.

(4)$^\star$ This exercise discusses dual varieties of possibly singular hypersurfaces. Let $X = \mathbb{V}(F) \subset \mathbb{P}^{n+1}$ be a degree $d$ irreducible hypersurface. Let $J \subset X$ be the closed subscheme

$$\mathbb{V}\left(\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_{n+1}}\right).$$

Let $\tilde{X} = \mathrm{Bl}_J X$ with $\pi : \tilde{X} \to X$.

(a) Describe a natural map $g : \tilde{X} \to \hat{\mathbb{P}}^{n+1} \times X$, where $\hat{\mathbb{P}}^{n+1}$, as in Sheet 2, parametrises the hyperplanes of $\mathbb{P}^n$. Let $f : \tilde{X} \to \hat{\mathbb{P}}^{n+1}$ be the composition of the $g$ with the projection. The reduced image of $f$ is the *dual variety* of $X$.

(b) Note that $\tilde{X} = \mathrm{Bl}_J X$ carries a tautological line bundle $\mathscr{O}_{\tilde{X}}(-1)$ associated to the exceptional divisor $\mathbb{P}(C_J X)$. Let $s_i$ be the degree $i$ component of $s(J, X)$. Assuming the fact that $f^*\mathscr{O}_{\hat{\mathbb{P}}^{n+1}}(1) = \pi^*\mathscr{O}_X(d-1) \otimes \mathscr{O}_{\tilde{X}}(1)$, prove that

$$\deg f_*[\tilde{X}] = d(d-1)^n - \sum_{i=0}^{n} \binom{n}{i} (d-1)^i \deg(s_i).$$

(c) Specialise to the case where $X \subset \mathbb{P}^2$ is an irreducible plane curve of degree $d$, in which case the formula becomes

$$\deg f_*[\tilde{X}] = d(d-1) - \sum_{p \in X} e_J(X)_p.$$

Let $\nu : X' \to X$ be the normalisation and let $J' = \nu^{-1}(J)$. By considering proper pushforward of Segre classes, prove that $(e_J X)_p = \sum_{q \in \nu^{-1}(p)} \mathrm{ord}_q(J')$.

(d) Using the above formulas, compute $\deg f_*[\tilde{X}]$ for $X = \mathbb{V}(y^2 z - x^3)$.

# Intersection Theory Sheet IV

[*The questions on this sheet are all non-examinable.*]

(1) Let $E$ be a vector bundle of rank $r$ on a non-singular variety $Y$ of dimension $n$, let $X = \mathbb{P}(E)$, with projection $p : X \to Y$. Prove that

$$A^*(X) \cong A^*(Y)[\xi]/(\xi^r + c_1(E)\xi^{r-1} + \cdots + c_r(E))$$

as graded rings, where $\xi = c_1(\mathscr{O}_E(1))$.

(2) Let $Y$ be non-singular, $y_i \in A^*Y$, $U_i \subset Y$ open subschemes such that $y_i$ restricts to 0 in $A^*U_i$, for $i = 1, \ldots, r$. Prove that $y_1 \cdot \cdots \cdot y_r \in A^*(Y)$ restricts to zero in $A^*(U_1 \cup \cdots \cup U_r)$. Deduce that if the $U_i$ cover $Y$, then $y_1 \cdot \cdots \cdot y_r = 0$.

(3) (a) Let $s$ (resp. $t$) be the class of a hyperplane on $\mathbb{P}^n$ (resp. $\mathbb{P}^m$), prove that

$$A^*(\mathbb{P}^n \times \mathbb{P}^m) = \mathbb{Z}[s,t]/(s^{n+1}, t^{m+1}).$$

(b) Let $H_i$, $i = 1, \ldots, n+m$ be hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^m$ of bidegree $(a_i, b_i)$. Using the fact that $[H_i] = a_i s + b_i t$, prove that

$$\int_{\mathbb{P}^n \times \mathbb{P}^m} [H_1] \cdots [H_{n+m}] = \sum_{\substack{\sigma \subset \{1,\ldots,n+m\},\\ \#\sigma = n}} \prod_{j \in \sigma} a_j \cdot \prod_{k \in \sigma^c} b_k.$$

(4) Let $X, Y$ be non-singular varieties.

(a) Prove that the exterior product gives a ring homomorphism of graded rings

$$\times : A^*X \times A^*Y \to A^*(X \times Y).$$

(b) If $X$ has a cellular decomposition, prove that the ring homomorphism is surjective.

(c) If $X = \mathbb{P}^r$, prove that it is an isomorphism.

(5)$^\star$ This exercise computes the Chow ring $A^*(\mathrm{Gr}_d(E))$, where $\mathrm{Gr}_d(E)$ is the Grassmannian of $d$-planes in an $n$-dimensional vector space $E$. Let $\mathcal{Q}$ be the universal quotient bundle on $\mathrm{Gr}_d(E)$: $\mathcal{Q}$ is a rank-$(n-d)$ vector bundle which fits in the short exact sequence

$$0 \to \mathcal{S} \to \mathscr{O}_{\mathrm{Gr}_d(E)}^{\oplus n} \to \mathcal{Q} \to 0.$$

Given a tuple $\lambda = (\lambda_1, \ldots, \lambda_d)$ with $\lambda_1 \geq \cdots \geq \lambda_d \geq 0$, set

$$\Delta_\lambda := \Delta_{\lambda_1,\cdots,\lambda_d}(c(\mathcal{Q})) := \det \begin{pmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+d-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & \\ & \cdots & \cdots & \\ c_{\lambda_d-d+1} & \cdots & \cdots & c_{\lambda_d} \end{pmatrix} \in A^*(\mathrm{Gr}_d(E)).$$

Here $c_i$ denotes $c_i(\mathcal{Q})$, and the matrix has entry $(i,j)$ equal to $c_{\lambda_i+(j-i)}$.

Given $\underline{a} = (a_1, \ldots, a_d)$ where $0 < a_1 < \cdots < a_d \leq n = \dim(E)$, define $A_i := \mathrm{Span}\{\mathbf{e}_j \mid j \leq a_i\} \subset E$. Recall that there are closed subvarieties

$$\Omega(\underline{a}) = \{W \in \mathrm{Gr}_d(E) \mid \dim(W \cap A_i) \geq i \text{ for all } i = 1, \ldots, d\}$$

and locally closed subschemes

$$\Omega^\circ(\underline{a}) = \{W \in \Omega(\underline{a}) \mid \mathrm{pr} : W \to \mathrm{Span}\{\mathbf{e}_{a_1}, \ldots, \mathbf{e}_{a_d}\} \text{ is an isomorphism}\} \cong \mathbb{A}^{\sum_{i=1}^d a_i - d}$$

defined in lectures, such that $\{[\Omega(\underline{a})]\}_{\underline{a}}$ forms an additive basis of the Chow group. Let $\lambda_i := n-d+i-a_i$, then *Giambelli's formula* states that in $A_*(\mathrm{Gr}_d(E))$,

$$[\Omega(\underline{a})] = \Delta_\lambda \cap [\mathrm{Gr}_d(E)].$$

(a) Using Giambelli's formula, prove that $c_i(\mathcal{Q})$ generates $A^*(\mathrm{Gr}_d(E))$ as a ring.

(b) Recall that $\mathcal{S}$ is a vector bundle of rank $d$. By considering the Chern classes of $\mathcal{S}$, describe a way of obtaining relations among $c_i(\mathcal{Q})$ that are homongeneous in degrees $d+1,\dots,n$. Write down the relations explicitly when $d=2, n=4$.

(c) Let $I$ be the ideal generated by the relations obtained in the previous part. It is a fact that the surjection $\mathbb{Z}[c_i(\mathcal{Q}) \mid i = 1,\dots n-d]/I \to A^*(\mathrm{Gr}_d(E))$ is always an isomorphism. By calculating the ranks on both sides, verify this when $d=2, n=4$. (Hint: using the basis $[\Omega(\underline{a})]$, prove that $A_*(\mathrm{Gr}_d(E))$ has rank $\binom{n}{d}$ as an abelian group.)