Part III Commutative Algebra

Example Sheet IV, 2021

Note: If you would like to receive feedback, please turn in solutions to Questions 3, 8, and 14 by noon on January 21st, at which time solutions will be posted. You turn your work in via the moodle course page.

1. Let A be a ring, $S \subseteq A$ a multiplicatively closed subset, and N, M A-modules with M *finitely presented*, i.e., there exists an exact sequence with n_1, n_2 non-negative integers

$$A^{n_1} \to A^{n_2} \to M \to 0.$$

(Note that if A is Noetherian, this is equivalent to M being finitely generated; otherwise it is stronger.) Show that there is an isomorphism of $S^{-1}A$-modules

$$S^{-1}\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

2. (a) Show that a UFD (Unique Factorization Domain) is integrally closed in its field of fractions, i.e., is normal.

   (b) Let k be a field and $A = k[x, y]/(y^2 - x^3)$. Is A normal? If not, what is the integral closure of A in its field of fractions?

3. Let $A \subseteq B$ be a subring of B, with the property that the complement of A in B, $B \setminus A$, is multiplicatively closed. Show that A is integrally closed in B.

4. Here are several more forms of Hilbert's Nullstellensatz. Let $A = k[x_1, \ldots, x_n]$ for k an algebraically closed field. For an ideal $I \subseteq A$, define

$$Z(I) := \{(a_1, \ldots, a_n) \in k^n \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\},$$

and for $X \subseteq k^n$ a subset, define

$$I(X) := \{f \in A \mid f(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in X\}.$$

   (a) Show that if $I \neq A$, then $Z(I)$ is non-empty.

   (b) Show that $I(Z(I)) = \sqrt{I}$.

5. Let $(A, \mathfrak{m})$, $(B, \mathfrak{n})$ be two local rings. We say B *dominates* A if A is a subring of B and $\mathfrak{m} = A \cap \mathfrak{n}$, or equivalently, $\mathfrak{m} \subseteq \mathfrak{n}$. Let K be a field and let Σ be the set of all local subrings of K. If Σ is ordered via the relation of domination, show Σ has maximal elements and that $A \in \Sigma$ is maximal if and only if A is a valuation ring of K.

6. Let A be a valuation ring of a field K. Let U be the group of invertible elements of A; this is a subgroup of $K^* = K \setminus \{0\}$, so that $\Gamma := K^*/U$ is also a group.

   (a) For $\bar{x}, \bar{y} \in \Gamma$ represented by $x, y \in K^*$, write $\bar{x} \geq \bar{y}$ if $xy^{-1} \in A$. Show that this defines a total ordering on Γ which respects the group structure, i.e., if $\bar{x} \geq \bar{y}$ then $\bar{x}\bar{z} \geq \bar{y}\bar{z}$.

   (b) Let $v : K^* \to \Gamma$ be the quotient map. Show that $v(x+y) \geq \min(v(x), v(y))$ for all $x, y \in K^*$ with $x + y \neq 0$.

7. Let K be a field, Γ a totally ordered abelian group (written additively). In other words, Γ is an abelian group with a total ordering respecting the group structure as above. A *valuation of K with values in* Γ is a mapping $v : K^* \to \Gamma$ satisfying the following properties:

   (a) $v(xy) = v(x) + v(y)$ (i.e., v is a group homomorphism).

   (b) $v(x + y) \geq \min(v(x), v(y))$ provided $x + y \neq 0$.

Show that

$$A := \{x \in K^* \mid v(x) \geq 0\} \cup \{0\}$$

is a valuation ring of K, with maximal ideal

$$\mathfrak{m} := \{x \in K^* \mid v(x) > 0\} \cup \{0\}.$$

So one speaks interchanageably of valuations and valuation rings.

8. Let k be a field, and Γ any totally ordered abelian group. Let $A := k[\Gamma]$ denote the group ring of Γ, i.e., $k[\Gamma]$ is the k-vector space with basis $\{z^\gamma \mid \gamma \in \Gamma\}$ and multiplication determined by

$$z^\gamma \cdot z^{\gamma'} = z^{\gamma+\gamma'}.$$

Show that A is an integral domain.

Define $v_0 : A \setminus \{0\} \to \Gamma$ by

$$v_0(\sum_{i\in I} \alpha_i z^{\gamma_i}) = \min\{\gamma_i \mid i \in I\}$$

where I is a finite index set and $\alpha_i \in k \setminus \{0\}$ for each i. Show v_0 satisfies conditions (a) and (b) of Question 7.

Now let K be the field of fractions of A. Show that v_0 can be extended to $v : K^* \to \Gamma$ so that v is a valuation with value group Γ.

9. Take $\Gamma = \mathbb{R}$ in the previous problem. Consider the ideal $I \subseteq K[x, y]$ given by $I = (1+x+y)$, and let $Z(I) \subseteq K^2$ be as defined in Question 4. Determine the image of $Z(I) \cap (K^*)^2$ under the map

$$(K^*)^2 \to \mathbb{R}^2, \qquad (a, b) \mapsto (v(a), v(b)).$$

10. Let K be a field, $A \subseteq K$ a valuation ring of A. Show that A is Noetherian if and only if either $A = K$ or the value group of A is $\mathbb{Z}$. In this case we say A is a *discrete valuation ring.*

[Note: After doing this problem, strongly I recommend reading Proposition 9.2 of Atiyah&MacDonald, which tells us how wonderful discrete valuation rings are. This is crucial in both algebraic geometry (for the study of divisors) and algebraic number theory.]

11. Let $A = A_1 \times A_2$ be a direct product ring. Show that the modules $M_1 = A_1 \times 0$, $M_2 = 0 \times A_2$ are projective. As they are not free, this shows that projective modules need not be free. [Note: there are many more less trivial examples. For example, if you are an algebraic number theorist, if $\mathcal{O}_K$ is a number ring, and I a fractional ideal which is not principal, then I is projective but not free.]

12. Let $A = k[x, y]/(xy)$, and let $\mathfrak{m} = (x, y)$. Construct a resolution of $A/\mathfrak{m}$ by finitely generated free modules. Use this resolution to calcuate $\mathrm{Ext}^i_A(A/\mathfrak{m}, A)$ and $\mathrm{Tor}^A_i(A/\mathfrak{m}, A/\mathfrak{m})$.

13. Let $S = k[x^3, x^2y, xy^2, y^3] \subseteq k[x, y]$, let $\mathfrak{m} = (x^3, x^2y, xy^2, y^3) \subseteq S$. Show that $S_\mathfrak{m}$ is Cohen-Macaulay by exhibiting an explicit regular sequence. You may use without proof that $\dim S_\mathfrak{m} = 0$.

14. Let $(A, \mathfrak{m})$ be a local ring, $k = A/\mathfrak{m}$, and let $f : M \to N$ be an A-module homomorphism between finitely generated modules. We say f is *minimal* if the induced map

$$f \otimes \mathrm{id} : M \otimes_A k \to N \otimes_A k$$

is an isomorphism.

(a) Show f is minimal if and only if f is surjective and $\ker f \subseteq \mathfrak{m}M$.

(b) Show that for any finitely generated A-module M, there exists a minimal homomorphism $f : F \to M$ with F free.

(c) Suppose

$$0 \longrightarrow K \xrightarrow{f} F \xrightarrow{g} M \longrightarrow 0$$

is exact with g minimal and K and F free. Then show the homomorphisms

$$f_* : \mathrm{Ext}^i_A(k, K) \to \mathrm{Ext}^i_A(k, F)$$

induced by f vanish. [Note: You may use the following. If $K = A^n$, $F = A^m$, and f is represented by an $n \times m$ matrix (c_{ij}), then f_* is given by the same matrix now viewed as giving a map of A-modules $(\mathrm{Ext}^i_A(k, A))^n \to (\mathrm{Ext}^i_A(k, A))^m$.]

15. Let $(A, \mathfrak{m})$ be a local ring, $k = A/\mathfrak{m}$. Let $F^\bullet \to M \to 0$ be a free resolution of a finitely generated module M by finitely generated free modules, with maps $f_i : F_{i+1} \to F_i$. We say this free resolution is *minimal* if all the maps $f_i \otimes \mathrm{id} : F_{i+1} \otimes_A k \to F_i \otimes_A k$ are zero. Show that if A is Noetherian, then minimal resolutions exist.

16. Let $(A, \mathfrak{m})$ be a Noetherian local ring, $k = A/\mathfrak{m}$, M a finitely generated A-module. We define the *projective dimension* $\mathrm{pd}_A M$ of M to be the smallest n for which there exists a free resolution

$$0 \to F_n \to \cdots \to F_0 \to M \to 0.$$

Note this need not be finite. Show that if $\mathrm{pd}_A M$ is finite, then it is the length of every minimal free resolution of M. Further, $\mathrm{pd}_A M$ is the smallest integer i for which $\mathrm{Tor}^A_{i+1}(k, M) = 0$.

17. Let A be a Noetherian local ring and M a finitely generated A-module with $\mathrm{pd}_A M$ finite. Show that

$$\mathrm{pd}_A M + \mathrm{depth}(M) = \mathrm{depth}(A).$$

[Hint: first do the case of $\mathrm{pd}_A M = 0$ (easy) and $\mathrm{pd}_A M = 1$ (use Q14). Then go by induction.]