Part III Commutative Algebra

Example Sheet I, 2021

Note: If you would like to receive feedback, please turn in solutions to Questions 4, 7, 8 and 9 by noon on the day of the first Examples Class, at which time solutions will be posted. You turn your work in via the moodle course page.

1. *The Chinese remainder theorem.* We say two ideals I, J of a ring A are *coprime* if $I + J = (1)$. Given ideals $I_1, \ldots, I_n$ of A, there is a natural map

$$\phi : A \to \prod_{i=1}^{n} A/I_i$$

defined by $\phi(a) = (a + I_1, \ldots, a + I_n)$. Show the following:

   (a) If I_i, I_j are coprime whenever $i \neq j$, then $\prod_{i=1}^n I_i = \bigcap_{i=1}^n I_i$. [Note the ideal product IJ is the ideal generated by elements of the form ab with $a \in I$, $b \in J$.]

   (b) ϕ is surjective $\Leftrightarrow$ I_i, I_j are coprime whenever $i \neq j$.

   (c) ϕ is injective $\Leftrightarrow$ $\bigcap_{i=1}^n I_i = 0$.

2. Let A be a ring, $I \subseteq A$ an ideal, M an A-module. Show that

$$(A/I) \otimes_A M \cong M/IM.$$

3. Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) = 0$ if m, n are coprime.

4. Calculate the following tensor products of rings, where k is a field:

   (a) $k \otimes_{k[x,y]} k[u,v]$, where the map $k[x,y] \to k$ is given by $x, y \mapsto 0$, and the map $k[x,y] \to k[u,v]$ is given by $x \mapsto u, y \mapsto uv$, with both maps the identity on k.

   (b) $k[v] \otimes_{k[x]} k[v]$, where both maps $k[x] \to k[v]$ are given by $x \mapsto v^2$, and are the identity on k.

5. Let $A[x]$ denote the polynomial ring in one variable over a ring A. Show that $A[x]$ is a flat A-module.

6. Some properties of localization:

   (a) Let $S \subseteq A$ be a multiplicatively closed subset of a ring A, M a finitely generated A-module. Show that $S^{-1}M = 0$ if and only if there exists an $s \in S$ such that $sM = 0$.

   (b) Let A be a ring, $S, T \subseteq A$ two multiplicatively closed subsets, and let U be the image of T in $S^{-1}A$. Show that the rings $(ST)^{-1}A$ and $U^{-1}(S^{-1}A)$ are isomorphic, where

$$ST = \{s \cdot t \,|\, s \in S, t \in T\}.$$

   (c) Let $f : A \to B$ be a ring homomorphism and let S be a multiplicatively closed subset of A. Let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.

7. Prove that

$$\left(\frac{k[x,y,z]}{(xy - z^2)}\right)_x \cong k[x,z]_x,$$

where the subscript x denotes localization at x.

8. Let A be an integral domain and M an A-module. An element $m \in M$ is *torsion* if there exists $a \in A \setminus \{0\}$ with $am = 0$. Show the set of torsion elements of M, written as $T(M)$, is a submodule of M. If $T(M) = 0$, we say M is *torsion free.* Show furthermore:

   (a) If M is any A-module, then $M/T(M)$ is torsion-free.

   (b) If $f : M \to N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.

   (c) If $0 \to M_1 \to M_2 \to M_3$ is an exact sequence of A-modules, then so is $0 \to T(M_1) \to T(M_2) \to T(M_3)$.

   (d) If $S \subseteq A$ is a multiplicatively closed subset, show that $T(S^{-1}M) = S^{-1}(TM)$ as $S^{-1}A$-modules.

9. Let A be a ring, and let F be the free A-module A^n. Show that every set of n generators $x_1, \ldots, x_n$ of F is a basis of F, i.e., whenever $\sum_i a_i x_i = 0$, we have $a_i = 0$ for all i. [Hint: First reduce to the case that A is a local ring, then use Nakayama's lemma to reduce to the case of vector spaces. You may use the following fact, which will hopefully be explained later in the course: If $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact and M_3 is flat, then for any A-module N, $M_1 \otimes_A N \to M_2 \otimes_A N$ is injective.]

10. Let A be a ring. We define

$$\operatorname{Spec} A := \{\mathfrak{p} \subset A \,|\, \mathfrak{p} \text{ is a prime ideal}\}.$$

This is called the *spectrum* of the ring A. For $I \subseteq A$ an ideal, we define

$$V(I) := \{\mathfrak{p} \in \operatorname{Spec} A \,|\, I \subseteq \mathfrak{p}\}.$$

(a) Show that the sets $V(I)$ form the closed sets of a topology on $\operatorname{Spec} A$. This topology is known as the *Zariski topology* on $\operatorname{Spec} A$.

(b) Let A be a ring. Show the sets

$$D(f) := \{\mathfrak{p} \in \operatorname{Spec} A \,|\, f \notin \mathfrak{p}\}$$

with f ranging over elements of A form a basis of the topology on $\operatorname{Spec} A$.

(c) If $\varphi : A \to B$ is a homomorphism of rings, show that if $\mathfrak{p} \in \operatorname{Spec} B$, then $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of A. Thus φ induces a function $\varphi^* : \operatorname{Spec} B \to \operatorname{Spec} A$. Show that φ^* is continuous with respect to the Zariski topology.

11. Let A be a ring. Show the following are equivalent:

i) $\operatorname{Spec} A$ is disconnected.

ii) There exists nonzero elements $e_1, e_2 \in A$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1 + e_2 = 1$.

iii) A is isomorphic to $A_1 \times A_2$ for some rings A_1, A_2.

12. *More about the spectrum.*

(a) Let A be a ring, S a multiplicatively closed subset of A, and $\varphi : A \to S^{-1}A$ the canonical homomorphism. Show that the induced map $\varphi^* : \operatorname{Spec} S^{-1}A \to \operatorname{Spec} A$ is a homeomorphism onto its image in $X = \operatorname{Spec} A$. We write this image as $S^{-1}X$. Show that in the case where $S = \{1, f, f^2, \ldots\}$, $S^{-1}X = D(f)$.

(b) Let $\varphi : A \to B$ be a ring homomorphism, with induced map $\varphi^* : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$. Let $S \subseteq A$ be a multiplicatively closed subset. Show that $\varphi(S)$ is a multiplicatively closed subset of B, and that there is an induced morphism $S^{-1}A \to \varphi(S)^{-1}B$. [Note: Question 6(c) justifies writing $S^{-1}B$ for the latter ring, which we will now do.] Thus we obtain an induced map $\operatorname{Spec} S^{-1}B \to \operatorname{Spec} S^{-1}A$. Show this map agrees with the restriction of φ^* to $S^{-1}Y$ under the identification of $S^{-1}X, S^{-1}Y$ with $\operatorname{Spec} S^{-1}A, \operatorname{Spec} S^{-1}B$ of part (a). Show that $(\varphi^*)^{-1}(S^{-1}X) = S^{-1}Y$.

(c) In the same setup as in (b), let $I \subseteq A$ be an ideal and $J = I^e$ be its extension in B. Let $\bar{\varphi} : A/I \to B/J$ be the induced homomorphism. If $\operatorname{Spec}(A/I)$ is identified with $V(I)$ and $\operatorname{Spec}(B/J)$ is identified with $V(J)$, show $\bar{\varphi}^*$ agrees with the restriction of φ^* to $V(J)$.

(d) Let $\mathfrak{p} \subseteq A$ be a prime ideal. Take $S = A \setminus \mathfrak{p}$ in (b) and then reduce modulo $\mathfrak{p}A_{\mathfrak{p}}$ as in (c). Deduce that the subspace $(\varphi^*)^{-1}(\mathfrak{p})$ of Y is naturally isomorphic to $\operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = \operatorname{Spec}(k(\mathfrak{p}) \otimes_A B)$, where $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, the *residue field* of the local ring $A_{\mathfrak{p}}$.