Part III Algebraic Geometry

Example Sheet IV, 2024. Turn in 2,4,5 by noon on January 27th, 2025.

1. Let $X = \operatorname{Spec} A$ be affine, and let
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
be an exact sequence of quasi-coherent sheaves of $\mathcal{O}_X$-modules. Show that
$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$$
is exact. [You may freely use the following fact about quasi-coherent sheaves: if $\mathcal{F}$ is a quasi-coherent sheaf on a scheme X, and $U \subseteq X$ is open affine, $U = \operatorname{Spec} A$, then $\mathcal{F}|_U = \widetilde{M}$ for some A-module M. You may find a proof of this fact in Hartshorne II, §5, Proposition 5.4. Note II, Proposition 5.6 proves a more general statement.]

2. Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X, and suppose given a long exact sequence
$$0 \to \mathcal{F} \to \mathcal{F}^0 \xrightarrow{d^0} \mathcal{F}^1 \xrightarrow{d^1} \cdots.$$
Suppose further that $H^i(X, \mathcal{F}^j) = 0$ for all $i > 0$ and all j. Show that
$$H^i(X, \mathcal{F}) = \frac{\ker d^i : \Gamma(X, \mathcal{F}^i) \to \Gamma(X, \mathcal{F}^{i+1})}{\operatorname{im}\, d^{i-1}(X, \mathcal{F}^{i-1}) \to \Gamma(X, \mathcal{F}^i)}$$
for all i. Sheaves $\mathcal{G}$ on X with $H^i(X, \mathcal{G}) = 0$ for all $i > 0$ are called *acyclic*.

3. We define a *flabby* (*flasque* in french) sheaf on a topological space X to be a sheaf $\mathcal{F}$ such that for any inclusion $V \subseteq U$ of open sets of X, the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

   We will prove, for an open covering $\mathcal{U}$ of a topological space X and a flabby sheaf $\mathcal{F}$ on X, that $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.

   [Note: You may want to use Zorn's Lemma in what follows.]

   a) Show that if
   $$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$
   is an exact sequence of sheaves on X and $\mathcal{F}_1$ is flabby, then $\mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is surjective for any open set $U \subseteq X$.

   b) Show that if
   $$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$
   is an exact sequence of sheaves with $\mathcal{F}_1$ and $\mathcal{F}_2$ flabby, then $\mathcal{F}_3$ is flabby.

   c) Show that if
   $$0 \to \mathcal{F}_0 \to \cdots \to \mathcal{F}_{p-1} \to \mathcal{F}_p \to \mathcal{F}_{p+1} \to \cdots$$
   is a long exact sequence of flabby sheaves, then
   $$0 \to \mathcal{F}_0(U) \to \cdots \to \mathcal{F}_{p-1}(U) \to \mathcal{F}_p(U) \to F_{p+1}(U) \to \cdots$$
   is exact.

   d) If $\mathcal{F}$ is a sheaf on X, $U \subseteq X$ an open set, denote by (as usual) $\mathcal{F}|_U$ the sheaf on X defined by $\mathcal{F}|_U(V) = \mathcal{F}(V \cap U)$. For an open covering $\mathcal{U}$ of X, let
   $$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0, \ldots, i_p \in I} \mathcal{F}|_{U_{i_0} \cap \cdots \cap U_{i_p}}$$
   so that
   $$C^p(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{C}^p(\mathcal{U}, \mathcal{F})).$$
   Define boundary maps $\delta : \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$ in the same manner as was done for C^p. Show that the sequence
   $$0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \to \cdots$$
   is exact.

   e) Show that if $\mathcal{F}$ is flabby, so is $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$. Combine this fact with c) and d) to conclude that $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p > 0$.

4. Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, $U = X \setminus \{(x, y)\}$ (removing the maximal ideal corresponding to the origin). By choosing a suitable affine cover of U, show that $H^1(U, \mathcal{O}_U)$ is naturally isomorphic to the infinite dimensional k-vector space with basis $\{x^i y^j \mid i, j < 0\}$. Thus in particular U is not affine.

5. Let X be a subscheme of $\mathbb{P}^2_k$ defined by a single homogeneous polynomial $f(x_0, x_1, x_2) = 0$. Assume that $(1, 0, 0) \notin X$. Then show X can be covered by the two affine open subsets $U = X \cap D_+(x_1)$, $V = X \cap D_+(x_2)$. Now compute the Čech complex explicitly and show that

$$\begin{aligned} \dim H^0(X, \mathcal{O}_X) &= 1 \\ \dim H^1(X, \mathcal{O}_X) &= (d-1)(d-2)/2 \end{aligned}$$

where d is the degree of f.

If you are familiar with concept *genus* of a non-singular projective curve from Part II algebraic geometry, in fact $\dim H^1(X, \mathcal{O}_X)$ agrees with the genus, so you have calculated the genus of a plane curve.

6. (a) Let B be an A-algebra, M a B-module. An *A-derivation of B into M* is a map $d : B \to M$ such that (1) d is additive; (b) $d(bb') = bd(b') + b'd(b)$ for all $b, b' \in B$; (c) $d(a) = 0$ for all $a \in A$. The *module of relative differentials of B over A* is a B-module $\Omega_{B/A}$ equipped with a derivation $d : B \to \Omega_{B/A}$ which is universal, i.e., for any $d' : B \to M$ a derivation, there exists a B-module homomorphism $f : \Omega_{B/A} \to M$ such that $d' = f \circ d$.

We construct $\Omega_{B/A}$ as follows. Let $f : B \otimes_A B \to B$ be the A-algebra homomorphism given by $f(b \otimes b') = bb'$, $I = \ker f$. Set $\Omega_{B/A} = I/I^2$ and $d : B \to I/I^2$ given by $db = 1 \otimes b - b \otimes 1$. Show that $\Omega_{B/A}, d$ satisfies the above universal property.

(b) Calculate $\Omega_{A/k}$, where $A = k[x_1, \ldots, x_n]$. Show that this is the free A-module generated by symbols $dx_1, \ldots, dx_n$ and $d(f) = \sum_i (\partial f / \partial x_i) dx_i$.

(c) In lecture, we defined $\Omega_{X/Y}$ for a separated morphism $f : X \to Y$ to be the conormal sheaf of the closed immersion $\Delta : X \to X \times_Y X$. Show that if $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ with B an A-algebra, then $\Omega_{X/Y}$ is the sheaf associated to the B-module $\Omega_{A/B}$.

(d) Conclude that $\Omega_{\mathbb{A}^n_k/\operatorname{Spec} k}$ is the free rank n $\mathcal{O}_{\mathbb{A}^n}$-module generated by symbols $dx_1, \ldots, dx_n$.