Part III Algebraic Geometry

Example Sheet I, 2024

Note: If you would like to receive feedback, please turn in solutions to Questions 2, 9, 11 and 12 by Monday, November 4th at noon, at which time solutions will be posted. Work is turned in via moodle. The first examples class will be on Monday, November 4th, at 3:30 pm in MR9.

1. Let A be a ring. Show the sets
$$D(f) := \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$$
with f ranging over elements of A form a basis of the topology on $\operatorname{Spec} A$. Show that f is nilpotent (i.e., there exists an $n > 0$ such that $f^n = 0$) if and only if $D(f)$ is empty.

2. Describe the topological spaces $\operatorname{Spec} \mathbb{R}[x]$, $\operatorname{Spec} \mathbb{C}[x, y]$, $\operatorname{Spec} \mathbb{Z}[x]$ and $\operatorname{Spec} \mathbb{F}_p[x]$, where $\mathbb{F}_p$ is the field with p elements.

3. Let A be a ring. Show the following are equivalent:

   i) $\operatorname{Spec} A$ is disconnected.

   ii) There exists nonzero elements $e_1, e_2 \in A$ such that $e_1e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1 + e_2 = 1$.

   iii) A is isomorphic to $A_1 \times A_2$ for some rings A_1, A_2.

4. Let $\mathcal{F}$ be a presheaf on a topological space. Complete the proof of the existence of the associated sheaf $\mathcal{F}^+$ by showing:

   1. $\mathcal{F}^+$ is a sheaf.
   2. There is a natural morphism of presheaves $\theta : \mathcal{F} \to \mathcal{F}^+$ such that the pair $(\mathcal{F}^+, \theta)$ satisfies the desired universal property.

   Further, show that $(\mathcal{F}^+)_p = \mathcal{F}_p$ for $p \in X$. Show that if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, there is an induced morphism $f^+ : \mathcal{F}^+ \to \mathcal{G}^+$ with $(f^+)_p = f_p$.

   Note the universal property can be rephrased in terms of the language of adjoint functors of category theory. Let $PreShv_X$ denote the category of presheaves of abelian groups on X and Shv_X the category of sheaves of abelian groups on X. There is an obvious inclusion functor $i : Shv_X \to PreShv_X$. The above construction tells us that i has an adjoint functor a given by $a(\mathcal{F}) = \mathcal{F}^+$. This means that for $\mathcal{F}$ a presheaf, $\mathcal{G}$ a sheaf, there are natural bijections $\operatorname{Hom}(\mathcal{F}, i(\mathcal{G})) \to \operatorname{Hom}(a(\mathcal{F}), \mathcal{G})$. Here, naturality means the obvious compatibility of these bijections with maps induced by maps $\mathcal{F}' \to \mathcal{F}$ and $\mathcal{G} \to \mathcal{G}'$.

5. Show that if $f : \mathcal{F} \to \mathcal{G}$ is a morphism between sheaves, then the sheaf image $\operatorname{im} f$ can be naturally identified with a subsheaf of $\mathcal{G}$.

6. Show that a sequence $\cdots \to \mathcal{F}_{i-1} \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to \cdots$ is exact if and only if for every $p \in X$, the corresponding sequence of maps of abelian groups is exact.

7. Show a morphism of sheaves is an isomorphism if and only if it is injective and surjective.

8. Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$. Show that the natural map $\mathcal{F} \to \mathcal{F}/\mathcal{F}'$ is surjective and has kernel $\mathcal{F}'$, so that there is an exact sequence
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}/\mathcal{F}' \to 0.$$
Conversely, if
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
is an exact sequence, show that $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$ and $\mathcal{F}''$ is isomorphic to the quotient of $\mathcal{F}$ by this subsheaf.

9. For any open subset $U \subseteq X$, show that the functor $\Gamma(U, \cdot)$ is left exact, i.e., given an exact sequence
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'',$$
we obtain an exact sequence
$$0 \to \Gamma(U, \mathcal{F}') \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{F}'')$$
of abelian groups.

10. f^{-1} *and* f_* *are adjoint functors.* Given a continuous map $f : X \to Y$, sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, construct natural maps $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{G} \to f_*f^{-1}\mathcal{G}$. Use this to construct a bijection

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \to \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}),$$

(i.e., f^{-1} is left adjoint to f_* and f_* is right adjoint to f^{-1} in the language of category theory.)

11. Show that if $f \in A$, $X = \operatorname{Spec} A$, then as locally ringed spaces, $(D(f), \mathcal{O}_X|_{D(f)}) \cong \operatorname{Spec} A_f$.

12. Let A be a ring, $(X, \mathcal{O}_X)$ a scheme. Show there is a bijection

$$\alpha : \mathrm{Hom}_{Sch}(X, \operatorname{Spec} A) \to \mathrm{Hom}_{Rings}(A, \Gamma(X, \mathcal{O}_X))$$

given by associating to a morphism $f : X \to \operatorname{Spec} A$ the induced map on global sections

$$f^\# : \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to \Gamma(X, \mathcal{O}_X).$$

13. Show that $\operatorname{Spec} \mathbb{Z}$ is a final object in the category of schemes, i.e., every scheme has a unique morphism to $\operatorname{Spec} \mathbb{Z}$.

14. *Gluing.* Let $\{X_i\}$ be a family of schemes (possibly infinite) and suppose for each $i \neq j$ we are given an open subscheme $U_{ij} \subseteq X_i$. Suppose also given for each $i \neq j$ an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$, such that (1) for each i, j, $\varphi_{ji} = \varphi_{ij}^{-1}$ and (2) for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$.

    Then show there is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i with an open subscheme of X; (2) the $\psi_i(X_i)$ cover X; (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$; and (4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.