Metric and Topological Spaces **Easter 2016**

Example Sheet 1

1. Show that the sequence $2015, 20015, 200015, 2000015\ldots$ converges in the 2-adic metric on $\mathbb{Z}$.

2. Determine whether the following subsets $A \subset \mathbb{R}^2$ are open, closed, or neither:

   (a) $A = \{(x, y) \mid x < 0\} \cup \{(x, y) \mid x > 0, y > 1/x\}$

   (b) $A = \{(x, \sin(1/x) \mid x > 0\} \cup \{(0, y) \mid y \in [-1, 1]\}$

   (c) $A = \{(x, y) \mid x \in \mathbb{Q}, x = y^n$ for some positive integer $n\}$.

3. Show that the maps $f, g : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x + y$ and $f(x, y) = xy$ are continuous with respect to the usual topology on $\mathbb{R}$. Let X be $\mathbb{R}$ equipped with the topology whose open sets are intervals of the form (a, ∞). Are the maps $f, g : X \times X \to X$ continuous?

4. Let $\mathbf{C}^1[0, 1] = \{f : [0, 1] \to \mathbb{R} \mid f$ is differentiable and f' is continuous$\}$. For $f \in \mathbf{C}^1[0, 1]$, define

$$\|f\|_{1,1} = \int_0^1 (|f(x)| + |f'(x)|)\, dx.$$

   Show that $\|\cdot\|_{1,1}$ defines a norm on $\mathbf{C}^1[0, 1]$. If a sequence (f_n) converges with respect to this norm, show that it also converges with respect to the uniform norm. Give an example to show that the converse statement does not hold.

5. Let $d : X \times X \to \mathbb{R}$ be a function which satisfies all the axioms for a metric space except that instead of demanding that $d(x, y) = 0 \Leftrightarrow x = y$ we only require that $d(x, x) = 0$ for all $x \in X$. For $x, y \in X$, define $x \sim y$ if $d(x, y) = 0$. Show that $\sim$ is an equivalence relation on X, and that d induces a metric on the quotient $X/\sim$.

6. Find a closed $A_1 \subset \mathbb{R}$ (with the usual topology) so that $\overline{\mathrm{Int}(A_1)} \neq A_1$ and an open $A_2 \subset \mathbb{R}$ so that $\mathrm{Int}(\overline{A_2}) \neq A_2$.

7. Let $f : X \to Y$ be a map of topological spaces. Show that f is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$. Deduce that if f is surjective and continuous, the image of a dense set in X is dense in Y.

8. Suppose X is a topological space and $Z \subset Y \subset X$. If Y is dense in X and Z is dense in Y (with the subspace topology), must Z be dense in X?

9. Define a topology on $\mathbb{R}$ by declaring the closed subsets to be those which are i) closed in the usual topology and ii) either bounded or all of $\mathbb{R}$. Show that this is a topology, that all points of $\mathbb{R}$ are closed with respect to it, but that the topology is not Hausdorff.

10. The *diagonal* in $X \times X$ is the set $\Delta_X = \{(x, x) \mid x \in X\}$. If X is a Hausdorff topological space, show that Δ_X is a closed subset of $X \times X$.

11. Exhibit a countable basis for the usual topology on $\mathbb{R}$.

12. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus. Let $L \subset \mathbb{R}^2$ be a line of the form $y = \alpha x$, where α is irrational, and let $\pi(L)$ be its image in T^2. What are the closure and interior of $\pi(L)$?

13. Let $A = \{(0, 0, 1), (0, 0, -1)\} \subset S^2$. Let $B \subset T^2$ be the image of $\mathbb{R} \times 0 \subset \mathbb{R}^2$, where we view $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Show that the quotient spaces S^2/A and T^2/B are homeomorphic.

14. Let $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be a function which satisfies all the axioms for a norm except possibly the triangle inequality. Let $B = \{\mathbf{v} \in \mathbb{R}^2 \mid \|\mathbf{v}\| \leq 1\}$. Show that $\|\cdot\|$ is a norm if and only if B is a convex subset of $\mathbb{R}^2$. (That is, if $\mathbf{v}_1, \mathbf{v}_2 \in B$, then $t\mathbf{v}_1 + (1-t)\mathbf{v}_2 \in B$ for $t \in [0, 1]$.) For $r \in (0, \infty)$, let $\|\mathbf{v}\|_r = (|v_1|^r + |v_2|^r)^{1/r}$. Use calculus to sketch B for different values of r. Deduce that $\|\cdot\|_r$ is a norm for $1 \leq r < \infty$, but not for $0 < r < 1$.

15. Let D^2 be the closed unit disk in $\mathbb{R}^2$, and let X be the complement of two disjoint open disks in D^2. Let Y be the complement of a small open disk in T^2 (viewed as $\mathbb{R}^2/\mathbb{Z}^2$). Is X homeomorphic to Y? Is $X \times [0, 1]$ homeomorphic to $Y \times [0, 1]$? (No formal proof is required, but try to give some geometric justification.)

16. Show that the set of piecewise linear functions is dense in $\mathbf{C}[0, 1]$ with the sup metric. By considering piecewise linear functions where each linear piece is given by an expression with rational coefficients, deduce that $\mathbf{C}[0, 1]$ has a countable dense subset.

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