Metric & Topological spaces, Sheet 1: 2006

1. (a) Let $(X, \mathcal{T}_X)$ be a discrete topological space and $(Y, \mathcal{T}_Y)$ an arbitrary topological space. Prove that every function $X \to Y$ is necessarily continuous.

   (b) Give a topology on $\mathbb{R}$ which is neither trivial nor discrete such that every open set is closed (and vice-versa).

2. Define a subset of the integers $\mathbb{Z}$ to be open either if it is empty or if for some $k \in \mathbb{Z}$ the set S contains all integers $\geq k$. Show this defines a topology. Is it metrisable ?

3. Let $\mathcal{T}$ be the topology on $\mathbb{R}$ for which open sets are ϕ, $\mathbb{R}$ and open intervals of the form $(-\infty, a)$. Show this is a topology, and describe the closure of the singleton set $\{a\}$. What are the continuous functions from $(\mathbb{R}, \mathcal{T})$ to $(\mathbb{R}, \mathcal{T}_{eucl})$?

4. Show that a function $f : M_1 \to M_2$ of metric spaces is continuous if it preserves limits of sequences, i.e. if for every sequence $(x_n) \subset M_1$ converging to $a \in M_1$, the sequence $(f(x_n)) \subset M_2$ converges to $f(a)$.

5. Give $\mathbb{R}$ the following topology: a subset H is closed in $\mathbb{R}$ if and only if H is closed in the usual (Euclidean) topology and also bounded (of finite length in the usual Euclidean distance). Show that this *is* a topology, that points are closed sets, but that this topology is not Hausdorff.

6. Which of the following are open in $(\mathbb{R}^2, \mathcal{T}_{eucl})$? [Draw pictures, detailed proofs are not needed.] (i) $\{y > x^2\}$ (ii) $\{y > x^2, y \leq 1\}$, (iii) $\{y > x^2, y \leq -1\}$.

7. A subset $A \subset X$ is *regularly open* if $A = int(cl(A))$ and *regularly closed* if $A = cl(int(A))$. For $\mathbb{R}$ with the usual topology

   (a) give an example of an open set which is not regularly open; (b) give an example of a closed set which is not regularly closed.

8. Prove or give a counterexample to the relations:

$$cl \left(\bigcup_{j \in J} A_j \right) = \bigcup_{j \in J} (cl(A_j)); \qquad cl \left(\bigcap_{j \in J} A_j \right) = \bigcap_{j \in J} (cl(A_j)).$$

   [Hint: consider the cases where the indexing set J is finite and infinite separately.] Do the same thing for the interior, rather than closure, operator.

9. Give a bounded open subset of $(\mathbb{R}, \mathcal{T}_{eucl})$ which is a not a finite union of open intervals.

10. Find a topological space X and a decomposition $X = Y \cup Z$ into disjoint topological subspaces Y, Z which are both dense subsets of X.

11. For a function $f : X \to Y$, show the following are equivalent:

   (a) f is continuous, i.e. the preimages of open sets are open. (b) The preimages of closed subsets of Y are closed in X. (c) For all $A \subset X$, $f(cl(A)) \subset cl(f(A))$.

12. Prove or give counterexamples to:

(a) A continuous function $f : X \to Y$ is an *open map* i.e. if $U \subset X$ is an open subset then $f(U)$ is an open subset of Y.

(b) If $f : X \to Y$ is continuous and bijective (that is, one-to-one and onto) then f is a homeomorphism.

(c) If $f : X \to Y$ is continuous, open and bijective then f is a homeomorphism.

13. Write a topological space $X = Y \cup Z$ as a union of (not necessarily disjoint) closed subsets. Prove that a function f on X is continuous if and only if $f|_Y$ and $f|_Z$ are continuous functions on Y, Z respectively, where Y, Z have the subspace topology induced from X. (The notation refers to the restriction of the function to the appropriate domain.)

14. Show that a space X may be homeomorphic to a subspace of a space Y whilst Y is homeomorphic to a subspace of X, but where X, Y are not themselves homeomorphic.

15. (i) Prove that the product of two metric spaces admits a metric which induces the product topology. (ii) Show that if the product of two metric spaces is complete, then so are the factors.

16. Let M denote the space of bounded sequences of real numbers with the *sup* metric. Show (i) the subspace of convergent sequences is complete and (ii) the subspace of sequences with only finitely many non-zero values is not complete.

17. Let M be a complete metric space and $f : M \to M$ be continuous. Suppose for some r the iterate $f^{\circ r} = f \circ \cdots \circ f$ (r times) is a contraction. Prove f has a unique fixed point in M.

18. Let $f, g : X \to Y$ be continuous functions where X is any topological space and Y is a Hausdorff topological space. Prove that $W = \{x \in X \mid f(x) = g(x)\}$ is a closed subspace of X. Deduce that the fixed point set of any continuous function on a Hausdorff space is closed.

19. Prove that in a Hausdorff space X, the intersection of all open sets containing a point $x \in X$ is just the singleton set $\{x\}$. If for a space Z we know that the intersection of the open sets containing z is just $\{z\}$ for each $z \in Z$, does it follow that Z is Hausdorff? Justify your answer.

20. Give the real line $\mathbb{R}$ the "scattered" topology $\mathcal{T}_S$ in which open sets are unions $U \cup V$ where U is open in the usual Euclidean topology and V is any subset of the irrationals. (Check this is a topology.)

(a) Prove $(\mathbb{R}, \mathcal{T}_S)$ is Hausdorff.

(b) Prove $(\mathbb{R}, \mathcal{T}_S)$ is not metrisable.

21. Prove or give counterexamples to the following: (a) If X, Y are topological spaces and $W \subset X \times Y$ is an open set in the product, then we may write $W = U \times V$ where U is open in X and V is open in Y. (b) If $f : X \to Y$ is a function from a space X to a set Y and we give Y the quotient topology, then f is continous.

22. Give an example of a pair of spaces X, Y which are not homeomorphic but for which $X \times I$ and $Y \times I$ are homeomorphic, where I denotes the unit interval with its usual topology.

23. Define an equivalence relation $\sim$ on the interval $[0, 1] \subset \mathbb{R}$ by $x \sim y \iff x - y \in \mathbb{Q}$. Show that the quotient space $I/\sim$ is indiscrete.

24. Let X, Y be topological spaces. Define an equivalence relation on $X \times Y$ by $(x, y) \sim (x', y') \iff y = y'$. Prove that $X \times Y/\sim$ is homeomorphic to Y.

25. Give your own example to show that if X is Hausdorff and Y is a quotient space of X then Y need not be Hausdorff.

26. Let C be the cylinder

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, \ 0 \le z \le 1\}.$$

Using a picture, or otherwise, describe the quotient spaces of C by the following two equivalence relations:

(a) $(x, y, z) \sim (x', y', z') \iff x = -x', \ y = -y', \ z = z'$.

(b) $(x, y, z) \sim (x', y', z') \iff x = -x', \ y = -y', \ z = 1 - z'$.

27. Let X be a *normal* topological space: given disjoint closed sets A, B in X we may separate them by disjoint open sets $U \supset A$, $V \supset B$. Assume moreover that the singleton points of X are closed sets. Let $W \subset X$ be closed. Prove that the quotient space X/W (quotienting by the equivalence relation which collapses W to a point) is Hausdorff.

28. Let G be a topological group; so G is a topological space and there are given a distinguished point $e \in G$, continuous functions $m : G \times G \to G$ and $i : G \to G$ (multiplication and inverse) which satisfy the (usual, algebraic) group axioms. (Typical examples are matrix groups like $SL_2(\mathbb{R})$, the unit circle in $\mathbb{C}$ etc.) Prove the following:

(a) G is homogeneous: given any $x, y \in G$ there is a homeomorphism $\phi : G \to G$ such that $\phi(x) = y$.

(b) If $\{e\} \subset G$ is a closed subset, then the diagonal $\Delta G = \{(g, g) \mid g \in G\} \subset G \times G$ is a closed subgroup of $G \times G$.

(c) If $\{e\}$ is closed in G then the centre $Z(G) = \{g \in G \mid gh = hg \ \forall h \in G\}$ is a closed normal subgroup of G.

(d) Let H be an algebraic subgroup of G. Give the set of cosets $(G : H)$ the quotient topology from the natural projection map $\pi : G \to (G : H)$. Prove that π is an open map (images of open sets are open). [NB The usual notation for $(G : H)$ is G/H but this clashes with our notation for "collapsing subsets to a point".]

(e) Prove $(G : H)$ is Hausdorff if and only if H is closed in G.

Ivan Smith
is200@cam.ac.uk